Sharp fronts and Almost-Sharp Fronts of A Singular Surface Quasi-Geostrophic Equation

by

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. I declare that this thesis has not been submitted in any previous application for any degree and it contains the material which is my own original work, unless otherwise stated, cited or commonly known.

Abstract

In this thesis, we generalise results on sharp fronts and almost-sharp fronts by Fefferman, Luli, and Rodrigo [67], [68], [28], [26], [27], [19] to a singular variant of the Surface Quasi-Geostrophic Equation (SQG), where the velocity $u = \nabla^{\perp}|\nabla|^{-1}\theta$ is replaced with the more singular velocity $\nabla^{\perp}|\nabla|^{-1+\alpha}\theta$, for $\alpha \in (0, 1)$.

First, we derive the contour dynamics equation for a sharp front from the definition of a weak solution to our singular variant of SQG.

Then, we prove the existence of analytic sharp fronts to the sharp front equation using the abstract Cauchy–Kowalevskaya Theorem. This result is analogous to the result of Fefferman and Rodrigo in [27], which was a key result for proving the existence of analytic almost-sharp fronts whose existence time does not depend on the thickness of the transition region δ. The existence time in Sobolev spaces is not expected to be uniform in δ for almost-sharp fronts.

For such almost-sharp fronts, we study their evolution by understanding how curves supported in their transition region are transported by the velocity $u = \nabla^{\perp}|\nabla|^{-1+\alpha}\theta$. This work generalises the result of [19] to our more singular equation.

Finally, we define a spine curve for the almost-sharp front analogously to the spine curve of SQG in the model where one space variable is periodised, defined in the work of Fefferman and Rodrigo. The spine evolves according to the sharp front equation modulo an $O(\delta^{2-\alpha})$ error. As this does not vanish as $\alpha \to 1$, this formally suggests that the equation is in some sense not degenerate in this limit.

Abbreviations

PDE	partial differential equation
CDE	contour dynamics equation
SQG	surface quasi-geostrophic (equation)
SF	sharp front
ASF	almost-sharp front

Chapter 1

Introduction

In this thesis, we study special solutions, termed 'sharp fronts' and 'almost-sharp fronts' to the following singular variants of the (inviscid) Surface Quasi-Geostrophic Equation (SQG) which can be written as the following active scalar transport equation for $\alpha \in (0,1)$,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp |\nabla|^{-1+\alpha} \theta. \end{cases} \tag{1.1}$$

The model is meaningful even for the larger range $\alpha \in (0,2]$, but our focus is on the range $\alpha \in (0,1)$. Here, $x \in \mathbb{R}^2, t \in \mathbb{R}$, $\nabla = \binom{\partial_1}{\partial_2}$ is the spatial gradient operator, $\nabla^\perp = \binom{-\partial_2}{\partial_1}$ is the perpendicular gradient, the function $\theta = \theta(x,t) \in \mathbb{R}$ is the unknown scalar, and $u = u(x,t)$ is the associated velocity with the Fourier multiplier $|\nabla|^{-1+\alpha} = \mathcal{F}^{-1}|\xi|^{-1+\alpha}\mathcal{F}$. We will refer to this family of equations as singular SQG.

1.1 Motivation and literature review

In order to motivate the work presented in this thesis, we must first understand the endpoint $\alpha = 0$ case of (1.1), which corresponds to the SQG equation. It is so named because it originates from the field of geophysics, where in the regime of small Rossby and Ekmann numbers, it describes 'frontogenesis': the generation of fronts in the atmosphere between regions of hot and cold air. As we are primarily focused on the mathematical properties of this equation, we refer the reader to [64] for further details on the geophysical meaning of the SQG equation.

Instead of deriving the SQG equation by physical considerations, we will arrive at the SQG equation as a model of the three-dimensional Euler equations. Many well-known partial differential equations are active scalar transport equations.

For example, in the following class of active scalar transport equations,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ \qquad u = \nabla^{\perp} T(\theta), \end{cases}$$

the well-known 2D incompressible Euler equation governing the evolution of ideal fluids is obtained by choosing $T = (-\Delta)^{-1} = |\nabla|^{-2}$, the usual two-dimensional SQG equation is obtained by choosing $T = |\nabla|^{-1}$, and the so-called α-patches that interpolate between the 2D Euler equation and SQG has $T = |\nabla|^{-1+\alpha}$, $\alpha \in (-1, 0)$. From this perspective, the singular SQG equation (1.1) is a natural generalisation of the α-patches to the range $\alpha \in (0, 1)$, in which the velocity is determined by a more singular kernel than the previously mentioned equations.

Vortex filaments and sharp fronts

One particular area of interest in the theoretical of fluids is the rigorous foundation for the study of vortex filaments in a 3D Euler flow, which is heuristically a mathematically simplified model of vortex tubes. Vortex tubes are flows that are vorticity-free, except in a thin tubular region. The vorticity $\omega = \nabla \times u$ is the curl of the velocity u, and it solves the (incompressible) Euler equations,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u &= \nabla p, \\ \qquad\quad \nabla \cdot u &= 0. \end{cases}$$

Here, $(u \cdot \nabla) u^i := \sum_j u^j \partial^j u^i$, p is the associated pressure, and $\partial_t u + (u \cdot \nabla) u$ is the material derivative. Taking the curl of this equation (using the identity $(u \cdot \nabla) u = \omega \times u + \frac{1}{2} \nabla(|u|^2)$) leads to the equation for ω called the vorticity equation,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega &= u \cdot \nabla \omega, \\ \qquad\quad \nabla \cdot \omega &= 0. \end{cases} \tag{1.2}$$

A vortex filament is the following natural mathematical idealisation of a vortex tube, where the vorticity is instead given by a vector-valued measure supported on some curve $\mathbf{X} \in \mathbb{R}^3$. Formally, it should satisfy the 'local induction equation' (also known as the binormal curvature law, or the vortex filament equation in the literature), which states that the curve evolves in the direction of its binormal[1] at a

[1] The binormal of a curve in $\mathbb{R}^3$ is a vector that at every point with non-zero curvature, together with the tangent and normal to the curve, forms an orthonormal frame in three dimensions.

rate proportional to its curvature:

$$\mathbf{X}_t = \kappa \mathbf{b}.$$

However, this equation has not yet been derived rigourously from the Euler equations. The main stumbling block seems to be the failure of the Biot–Savart law,

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y) \, \mathrm{d}y,$$

which allows you to invert the curl operator for divergence-free fields. The homogeneity -2 of the kernel from the Biot–Savart law in 3D does not allow for an L^2 velocity to be defined from a vorticity given by a measure on a curve C, say $w = \Gamma \delta_C \mathbf{T}$, where $\delta_C(U) = \mathrm{length}(C \cap U)$ and $\mathbf{T} \in \mathbb{R}^3$ is the unit tangent vector of C. Roughly speaking, this corresponds to having a singularity along a curve, so the integral behaves locally like an integral of $d(x, C)^{-2}$, which is analogous to integrating $1/|x|^2$ in dimension 2, so is not convergent. To see this, one can first prove this when C is a straight line, and for more general curves, Taylor expand near a point in C. This detailed calculation can be found in the paper of Callegari and Ting [8]. The reason this is a problem is that the natural spaces to study the Euler equation are L^2 based.

Regardless, the local induction equation has good qualitative agreement with experiments (e.g. correctly predicting interesting topological behaviour), and it remains an active field of research, see for instance the recent work of the various authors Banica and Vega [5], Jerrard, Smets, and Seis,[37], [38], Fefferman, Pooley, and Rodrigo [65], [29], Davila, del Pino, Musso, and Wei [20], to name just a few.

In the SQG case, one can take the perpendicular gradient $\nabla^\perp = \binom{-\partial_{x^2}}{\partial_{x^1}}$ of the SQG equation to obtain the following equation,

$$\partial_t \nabla^\perp \theta + u \cdot \nabla(\nabla^\perp \theta) = (\nabla^\perp \theta) \cdot \nabla u.$$

By comparing with the vorticity equation for 3D Euler (1.2), we see that at least at a symbolic level, the role of the vorticity is played by $\nabla^\perp \theta$, and the analogous Biot–Savart law $u = |\nabla|^{-1} \nabla^\perp \theta$ is

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \nabla^\perp \theta(y) \, \mathrm{d}y,$$

which makes sense even if $\nabla^\perp \theta$ is supported on a curve, since the kernel of $|\nabla|^{-1}$ has homogeneity -1 in dimension 2. Moreover, a function whose gradient is supported on a curve is given by a piecewise constant function. We are therefore naturally led to the concept of a sharp front solution, which is a weak solution of SQG that is the indicator of a set.

In fact, the similarity with 3D Euler is more than formal. The SQG equation forms an excellent model equation for the 3D Euler equation with many other striking similarities, as noticed by Constantin, Majda, and Tabak [16], [15]. These similarities include:

- Vortex lines in 3D Euler (integral curves of the vorticity) correspond to level sets in SQG.
- In both equations, the velocity is recovered from the vorticity by a kernel of homogeneity $1-d$, where d is the spatial dimension.
- The infinitesimal length of a vortex line, given by $|\omega|$ evolves by an equation $\frac{D|\omega|}{Dt} = \alpha|\omega|$, and the infinitesimal length of a level set in SQG, given by $|\nabla^{\perp}\theta$ is $\frac{D|\nabla^{\perp}\theta|}{Dt} = \alpha|\nabla^{\perp}\theta|$. In both cases, $\alpha = (S\xi)\cdot\xi$ where S is the symmetric part of ∇v and ξ is the unit vector pointing in the direction of the vorticity (either ω in 3D Euler or $\nabla^{\perp}\theta$ in SQG).
- The classical Beale–Kato–Majda criterion [6] states that if ω is a smooth solution of the 3D Euler equations (in vorticity form) with a maximal time of existence $T^* < \infty$, then necessarily $\int_0^t \|\omega\|_{L^\infty}\,\mathrm{d}s \to \infty$ as $t \to T^*$. There is a direct analogue for the SQG equation: as t approaches the maximal time T^*, $\int_0^t \|\nabla^{\perp}\theta\|_{L^\infty}\,\mathrm{d}s \to \infty$.

Details can also be found in [50] and [49].

The analogous object to vortex filaments in the setting of the SQG equation are sharp fronts, because if the 'vorticity' $\nabla^{\perp}\theta$ is a vector-valued measure supported on some simple closed curve z parameterising some boundary ∂A of a set A (in the distributional sense), then the solution is constant away from the curve, and therefore the solution has to be of the form $\theta = \mathbf{1}_{x\in A(t)}$ (up to adding and multiplying constants). Hence, $\theta(x,t) = \mathbf{1}_{x\in A(t)}$ can only have non-trivial evolution at the interface ∂A, and in fact the solution is completely characterised by this contour dynamics equation (CDE).

There are many examples of other CDEs that have been studied in fluid dynamics, related to the other aforementioned active scalar transport equations. For instance, the Birkhoff–Rott equation in the Muskat problem as studied in [10], [11], [17]. In particular, the paper [77] discusses theoretically and numerically a generalised Birkhoff–Rott CDE that also covers the sharp fronts considered in this thesis, as well as some other biological models as studied in [36], and the paper [47] is

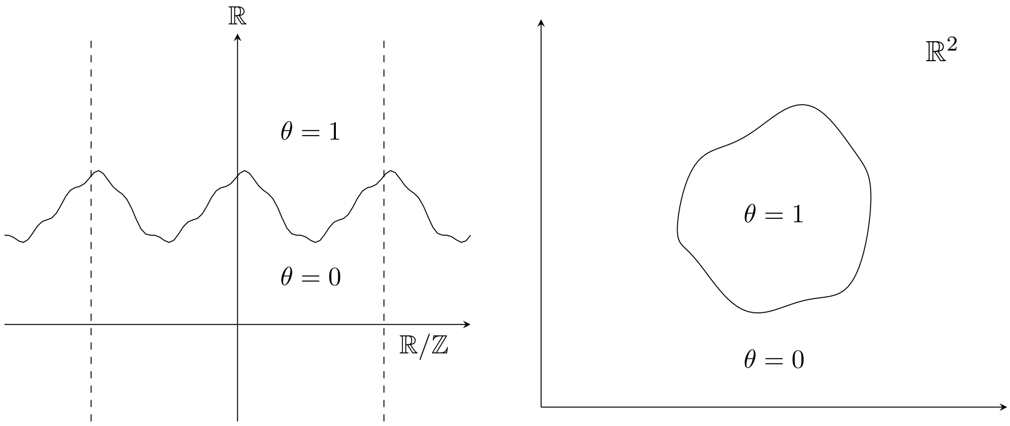

Figure 1.1: On the left: the sharp fronts obtained by periodising in one of the spatial variables, considered in the papers of Fefferman and Rodrigo [27], [67], [68]; on the right: the sharp fronts for a bounded domain considered in this thesis, and the work of Córdoba, Córdoba, and Gancedo [32], [18].

a modern review article on the topic. We remark that in the Birkhoff–Rott equation, the velocity is only discontinuous and not divergent as one approaches the interface, while the SQG equation and the generalised models considered in this paper are divergent in the direction of the tangent (similarly to the vortex filaments case). In spite of this, in all these scenarios, neglecting the evolution in the direction of the tangent leads to a well-defined CDE the curve.

Weak solutions of the SQG equation were first studied by Resnick in his thesis [66]. The study of sharp front solutions to SQG was initiated by Rodrigo in [68], where local existence and uniqueness was proved for C^∞ space-periodic graphs. The corresponding CDE for the Euler equation, termed the vortex patch problem, was first derived by Zabusky et. al. [87] and its systematic study can be found in the book [49]. Fefferman and Rodrigo [27] proved local existence for analytic graph data. Gancedo proved in [33], [32] and [18] existence and uniqueness of sharp fronts that are closed curves. The paper [42] gives local uniqueness and blowup for α-patches in certain ranges of $\alpha \in (-1, 0)$. The paper [13] gives the existence of sharp fronts for our equation (1.1) in Sobolev spaces. The paper [14] also discusses (1.1) and some other models. This thesis proves the local existence and uniqueness for sharp fronts for (1.1) that are closed curves ('sharp fronts for a bounded domain'), with initial curves that are analytic.

The two different settings (graph versus bounded domain) are morally the same, but have different technical tradeoffs. In the graph case, some calculations

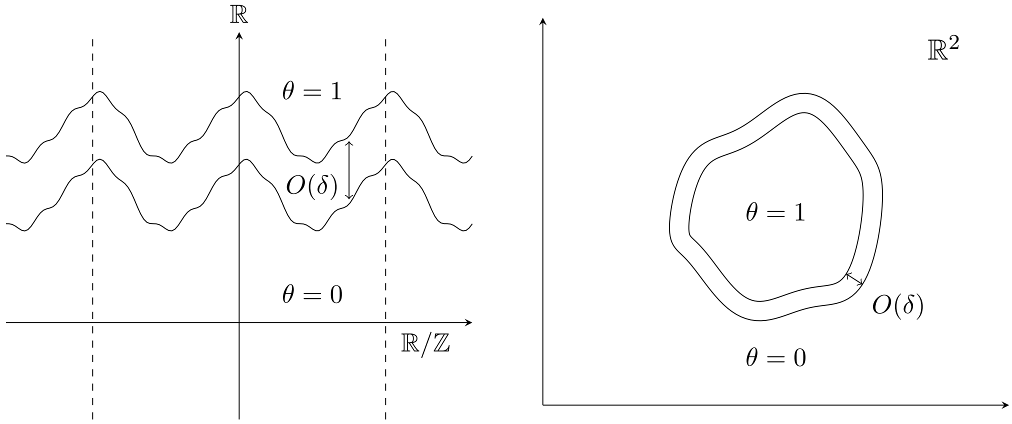

Figure 1.2: Almost-sharp fronts in both settings, analogous to Figure 1.1.

are more straightforward, but geometric quantities are not as simple to write down. Also, there is no inverse half-Laplacian for the space $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, so the equation has to be modified to take this into account. (However, no modification is needed for the Birkhoff–Rott equation.) In the case of the bounded domain, the equation need not be modified, but geometrical considerations make the discussion of almost-sharp fronts (further discussed below) slightly awkward, since we do not have 'natural' coordinates as in the case of the graph.

Almost-sharp Fronts as a model for thin regions of high vorticity

After the study of sharp fronts was initiated, Rodrigo, Fefferman, Córdoba, and Luli began to study almost-sharp fronts in a series of papers [19], [28], [28], [26], [30], which are, roughly speaking, smooth approximations to sharp fronts. We also mention the thesis [4] where almost-sharp fronts for the α-patches were studied. This is analogous to the study of a smooth solution to the vorticity equation whose vorticity is not the idealised one supported on a curve, but rather supported in some thin region around the filament, and so describes the more realistic fluids, termed 'vortex tubes' in the literature, that is modelled by the ideal vortex filament. This scenario has been studied in the original context of vortex filaments for the Euler equation, for instance in the (already mentioned) work of del Pino [20], and the hope is that the study of almost-sharp fronts of SQG (and the singular variants of this thesis) will lead to new insights in the theory of vortex filaments.

We have mentioned that for a sharp front $u = \mathbf{1}_{x \in A(t)}$, as x approaches the boundary $\partial A(t)$, the velocity diverges in the direction of the tangent to $\partial A(t)$.

So for an almost-sharp front, trajectories should move at an increasing speed as the thickness of the transition region shrinks. Despite this, Córdoba, Fefferman, and Rodrigo showed in [19] that for graph-like almost-sharp fronts of SQG, closed curves in the transition region are transported in a manner resembling the evolution equation of an SQG sharp front, and thus the geometry of an almost-sharp front is linked to the sharp front CDE, mirroring the known experimental results on vortex tubes. More precisely, if the gradient of an almost-sharp front was supported on a transition region of area $O(\delta)$, a curve inside this transition region transported by the velocity of an almost-sharp front of SQG was found to evolve by the same equation, up to $O(\delta \log \delta)$ errors.

Later, a further result was found in [26] for an intrinsically defined curve, which the authors termed a 'spine' for the almost-sharp front. Such a curve only had a $O(\delta^2 \log \delta)$ error in its evolution, so tracking the evolution of this special curve more accurately describes the evolution of the whole almost-sharp front than a generic compatible curve does.

In this thesis, we develop the analogous notion of an almost-sharp front for a bounded domain and prove that for the our singular variant of the SQG equation (1.1), the $O(\delta \log \delta)$ error rate is replaced by an $O(\delta^{1-\alpha})$ error rate. Then we construct a spine curve in our setting and prove that we also have the same improvement in the error rate by one power of δ to $O(\delta^{2-\alpha})$. Notably, the error rate for the spine does not degenerate as $\alpha \to 1$, which is when equation (1.1) formally degenerates to the trivial equation $\partial_t \theta = 0$. Therefore, even when the kernel $|x|^{-1-\alpha}$ is formally replaced by a kernel that is more singular, none of the error terms in the equation present new issues, which suggests that some non-trivial behaviour remains in the limit $\alpha \to 1$.

The above mentioned results of Rodrigo, Fefferman, Córdoba, and Luli culminated in the paper [30] of Fefferman and Rodrigo, where local existence of almost-sharp fronts for SQG was proven in the analytic class, with a time of existence T that did not depend on δ. This allows one to go back from an almost-sharp front to a sharp front, possibly even if one does not have a direct definition of a sharp front. If the analogous result could be proven for δ-thick vortex tubes around a vortex filament, this could be used to give a workable definition of a vortex filament solution to 3D Euler. The proof strategy that they employed is as follows:

1. Prove existence of an analytic sharp front This was carried out in [27].

2. Derive a well-behaved limit system for almost-sharp fronts using the analytic sharp front that does not depend badly on δ. This was carried out in [28].

3. Prove existence of analytic almost-sharp fronts. This was carried out in [30].

As already mentioned, this thesis successfully carries out the first step for our equation. We also derive a precise asymptotic equation for almost-sharp fronts of (1.1). However, it seems hard to reformulate the system in a way that the bad dependence in δ disappears. More precisely, the appearance of a logarithm in the approximate SQG equation separates a product of two bad terms into two pieces, each of which is manageable on its own. However, our approximate equation for (1.1) has a power law replacing the logarithm, and so both terms have to be dealt with at the same time. In particular, the methods employed in [28] and [30] do not seem like they can be extended to this scenario without significant new ideas. So in a sense, (1.1) displays features of both SQG sharp fronts and vortex filaments, since we can describe sharp fronts and almost-sharp fronts, but it seems that we need a renormalisation of some kind for the analogue of 'vortex tubes' to describe the analogue of a 'vortex filament'. This is discussed further in Sections 5.2 and 5.3 of Chapter 5, and in Chapter 7.

1.2 Outline of the thesis

The remainder of this thesis is organised as follows. First, in the rest of Chapter 1, we list some notation used throughout this thesis, and give a very brief overview on the geometry of planar curves.

In Chapter 2, we give the definition of a sharp front for singular SQG and derive its contour dynamics equation (CDE) from the definition of a weak solution,

$$\partial_t z \cdot N = \left(- \int_{s_* \in I} \frac{\partial_s z_* - \partial_s z}{|z - z_*|^{1+\alpha}} \, \mathrm{d}s_* \right) \cdot N.$$

In Chapter 3, we discuss and prove an abstract Cauchy–Kowalevskaya Theorem which proves that solutions to partial differential equations (PDEs) with a certain structure have solutions that are analytic in space, despite having differential operators of order higher than one. This is in contrast with the classical Cauchy–Kowalevskaya Theorem which gives also analyticity in time, but cannot be used for PDEs with operators of order higher than one.

In the Chapter 4, we reformulate slightly the above CDE for a sharp front, and carefully calculate the equations satisfied by the sharp front and its derivatives, following the scheme set out in [27]. We also introduce a function Γ that quantifies the 'arc-chord condition', which ensures that a curve is regular and does not self-intersect.

Here, we find the operator

$$\mathcal{H}_{1+\alpha}(h) := \int_{\mathbb{T}} \frac{h_s(s+s_*) - h_s(s_*)}{|\operatorname{Sin} s_*|^{1+\alpha}} \, \mathrm{d}s_*,$$

which is an operator of order higher than one. However, similarly to the heat equation, the solution operator $(\partial_t - \mathcal{H}_{1+\alpha})^{-1}$ is bounded on L^2 Sobolev spaces, and allows the use of the abstract Cauchy–Kowalevskaya Theorem to prove:

Theorem 1.1 (Existence for analytic sharp fronts, simplified statement)**.** *Let $z_0 : \mathbb{T} \to \mathbb{R}^2$ be an analytic curve $z_0 = z_0(s)$ with a regular parameterisation and no self-intersections. Then, there exists a unique solution $z = z(s,t)$ to the sharp-front CDE (4.1), defined for small times $t < T$ that is analytic in s for every t.*

In Chapter 5, we define an almost-sharp front (ASF) for singular SQG, and its 'compatible curves', which provide coordinates to describe the almost-sharp front in the thin transition region. (We omit a precise definition of compatible curves in this introduction). We then derive an asymptotic equation using the tubular neighbourhood coordinates by using the asymptotic result of Lemma 5.8. More precisely, if T, N are the basis vectors for the Frenet frame for the sharp front z, $L(\tau)$ is the length of z at time τ, κ is the curvature of z, and $\Omega(s,\xi,\tau) = \theta(z(s,\tau) + \xi\delta N(s,\tau), \tau)$ is the ASF expressed in the tubular coordinates (s,ξ), we prove the following theorem.

Theorem 1.2 (Approximate Equation for an ASF)**.** *Ω is a δ-ASF for singular SQG in the sense of Definition 5.1 iff in the tubular neighbourhood of the sharp front z, it solves the following approximate equation,*

$$\begin{aligned}
o(1) = \partial_\tau \Omega &- \frac{z_\tau \cdot T}{L} \partial_s \Omega \\
&+ (2+2\alpha) L \int_{-1}^{1} \int_{\mathbb{T}} \frac{T_* \cdot N}{|z - z_*|^{3+\alpha}} (z - z_*) \cdot (\xi N - \xi_* N_*) \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \int_{-1}^{1} \int_{\mathbb{T}} \frac{L \kappa_* T_* \cdot N}{|z - z_*|^{1+\alpha}} \xi_* \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \frac{C_{1,\alpha} \delta^{-\alpha}}{L} \int_{-1}^{1} \frac{\nabla \Omega_*^\perp |_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* \cdot \nabla \Omega \\
&+ \frac{C_{2,\alpha}}{L^{1+\alpha}} \int_{-1}^{1} \nabla \Omega^\perp |_{s_*=s} \, \mathrm{d}\xi_* \cdot \nabla \Omega \\
&+ \int_{-1}^{1} \int_{\mathbb{T}} \frac{T_* \cdot T \nabla^\perp \Omega_* \cdot \nabla \Omega}{|z - z_*|^{(1+\alpha)/2}} - \frac{\pi^{1+\alpha} \nabla^\perp \Omega_* |_{s_*=s} \cdot \nabla \Omega}{L^{1+\alpha} |\sin(\pi(s - s_*))|^{1+\alpha}} \, \mathrm{d}s_* \, \mathrm{d}\xi_*.
\end{aligned}$$

We finish this chapter with a small section showing that the function h which is defined by integrating across the transition region (that is, $h(s) := \int_{-1}^{1} \Omega(s,\xi)\,\mathrm{d}\xi$) can be used to simplify the asymptotic equation, while also having a limit equation as $\delta \to 0$. This function proved to be important in [28] and [30], where Fefferman and Rodrigo proved the existence of almost-sharp fronts with a time of existence independent of the parameter $\delta \ll 1$. However, despite the h equation having a well-defined limit equation, it is not enough to regularise the above approximate equation for Ω. This shows that a new idea or method is needed to achieve a similar existence result for almost-sharp fronts to the equation (1.1).

In Chapter 6, we study the evolution of almost-sharp fronts, by studying the evolution of their compatible curves, and a specially selected curve called a spine curve, which is equal to the compatible curve up to $O(\delta)$ adjustments. We show that these compatible curves of almost-sharp fronts solve the sharp front CDE in the weak sense, up to a small error $O(\delta^{1-\alpha})$. In [19], they show that compatible curves for a graph solve the SQG sharp front equation with the error rate $O(\delta \log \delta)$. Since $\log \delta$ is 'like' $\delta^{1-\alpha}$ for $\alpha = 1$, This thesis extends this result to the family of equations in $\alpha \in (0, 1)$, which we write as the following theorem:

Theorem 1.3 (Evolution of compatible curves)**.** *Suppose that θ is an ASF solution to (1.1), and z is a compatible curve. Then as z is transported by u, it evolves (in the weak sense) by the sharp front equation up to $O(\delta^{1-\alpha})$ errors,*

$$\partial_t z \cdot N = \left(-\int_{s_* \in I} K(z - z_*)(\partial_s z_* - \partial_s z)\,\mathrm{d}s_* \right) \cdot N + O(\delta^{1-\alpha}).$$

We also give an elementary proof for the following slightly weaker result:

Proposition 1.4. *Suppose that θ is an ASF solution to (1.1), and z is a compatible curve. Then as z is transported by u, for any $\epsilon > 0$, it evolves (in the weak sense) by the sharp front equation up to $O(\delta^{1-\alpha-\epsilon})$ errors,*

$$\partial_t z \cdot N = \left(-\int_{s_* \in I} K(z - z_*)(\partial_s z_* - \partial_s z)\,\mathrm{d}s_* \right) \cdot N + O(\delta^{1-\alpha-\epsilon}).$$

In order to obtain this result, we use the following lemma that can be seen as a fractional Leibniz rule for the product of a Hölder function and indicator function $\mathbf{1}_A$, with $s < s'$.

$$\|\Lambda^s (f \mathbf{1}_A)\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|f\|_{L^\infty(A)}^2 |A|^{1-2s} + [f]_{C^{s'}(A)}^2 |A|^{1-\frac{2(s-s')}{d}}.$$

In the second half of Chapter 6, we adapt the methods used in [26] to construct a spine for almost-sharp fronts, and derive its evolution equation. The formulation here differs slightly from [26] since our solutions of (1.1) are defined on $\mathbb{R}^2$ instead of $\mathbb{T} \times \mathbb{R}$.

Theorem 1.5 (Evolution of a spine)**.** *For an ASF solution to* (1.1)*, the spine curve S defined in Definition 6.5 evolves according to the sharp front equation up to $O(\delta^{2-\alpha})$ errors. That is,*

$$\partial_t S \cdot N_{out} = \left(\int_{s_* \in I} K(S - S_*)(\partial_s S_* - \partial_s S) \, \mathrm{d}s_* \right) \cdot N_{out} + O(\delta^{2-\alpha}).$$

In the final chapter, Chapter 7 we conclude by summarising the results of this thesis, and discuss some potential future research directions.

1.3 Notation

The following is a list of notation used throughout this thesis. The less standard notation is re-introduced when it appears in the text.

Functions and Spaces

We write '$f = f(a) \in Y$' to mean that f is a function with values in Y, typically written with the variable a. So if a denotes a typical element of a set A, then the function f is of the form $f : A \to Y$.

Typical such sets A that we use are Cartesian products of the natural numbers $\mathbb{N}$ (and the related set $\mathbb{N}_0 := \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$), the real numbers $\mathbb{R}$, the non-negative numbers $\mathbb{R}^+ := [0, \infty)$, the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, or subsets thereof. For the torus, we use either $I = [0, 1)$ or $I = [-1/2, 1/2)$ as a fundamental domain, by which we mean that all our expressions can be thought of as expressions defined initially on I, and are then extended periodically with period 1. If we say that for instance, a function $f : \mathbb{T} \to \mathbb{R}$ is smooth, then we mean that the extension (which is the 1-periodic function defined on $\mathbb{R}$ in the above manner) is smooth. For instance, if $x \in \mathbb{T}$ with fundamental domain $[-1/2, 1/2)$, then the function $|x|$ is continuous.

Multi-indices

An (n-dimensional) multi-index is an n-tuple,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$

We define the partial order

$$\alpha \geq \beta \iff \alpha_i \geq \beta_i \text{ for each } i \in \{1, \dots, n\}.$$

Addition and subtraction (as long as $\alpha \geq \beta$) of n-dimensional multi-indices is defined component-wise,

$$\alpha \pm \beta := (\alpha_1 \pm \beta_1, \dots, \alpha_n \pm \beta_n).$$

In addition, we define the following symbols for $\beta \leq \alpha$:

$$\begin{aligned}
|\alpha| &:= \sum_{i=1}^{n} \alpha_i, \\
\alpha! &:= \prod_{i=1}^{n} \alpha_i! \\
\binom{\alpha}{\beta} &:= \frac{\alpha!}{\beta!(\alpha-\beta)!}, \\
x^\alpha &:= \prod_{i=1}^{n} x_i^{\alpha_i}, \\
D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.
\end{aligned}$$

These definitions serve to simplify calculations when working in dimensions $n > 1$. For example, the multivariable Leibniz rule for smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where the sum is over all multi-indices β in $\mathbb{N}_0^n$, such that $\beta \leq \alpha$ under the partial order defined above. Observe that the notation was set up to closely match the one dimensional Leibniz rule, $(fg)^{(n)} = \sum_{k=0}^{n} \binom{k}{n} f^{(k)} g^{(n-k)}$.

Geometry

For a vector $(a, b) \in \mathbb{R}^2$, we write $(a, b)^\perp = (-b, a)$ for its 90° anti-clockwise rotation. We also borrow the following notation from differential geometry. First, we will occasionally write the components of vectors with superscripts $v = (v^1, \dots, v^n)$ instead of subscripts $v = (v_1, \dots, v_n)$. Secondly, we will sometimes say that we are using the Einstein summation convention. By this, we mean that repeated indices in a term implies that there is a summation over that index.

Integrals and Integral Operators

If $A \subset \mathbb{R}^n$, and $f : A \to \mathbb{C}$, the symbols $\int_A f(x)\,\mathrm{d}x = \int_A f\,\mathrm{d}x = \int_A f$ are to be interpreted as integrals with respect to Lebesgue measure over the set A. If f takes values in $\mathbb{C}^m$, then the integral is understood component-wise:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad \int_A f\,\mathrm{d}x := \begin{pmatrix} \int_A f_1\,\mathrm{d}x \\ \vdots \\ \int_A f_m\,\mathrm{d}x \end{pmatrix}.$$

We will often use different measures; at each instance, we will make this explicit in the notation, writing e.g. $\int_A f\,\mathrm{d}\mu = \int_A f(x)\,\mathrm{d}\mu = \int_A f(x)\,\mathrm{d}\mu(x) = \int_A f(x)\,\mathrm{d}\mu(x)$, and point out in writing that μ is not Lebesgue measure. For instance, we will exclusively use $\mathrm{d}l$ for the measure of arc-length of a curve, and $\mathrm{d}\sigma$ for a surface measure.

We will frequently employ the shorthand notation (loosely borrowed from kinetic theory),

$$\int_A F(g, f_*)\,\mathrm{d}s_* := \int_A F(g(s), f(s_*))\,\mathrm{d}s_*,$$
$$\int_A \int_B F(g, f_*)\,\mathrm{d}s_*\,\mathrm{d}\xi_* := \int_A \int_B F(g(s,\xi), f(s_*, \xi_*))\,\mathrm{d}s_*\,\mathrm{d}\xi_*.$$

That is, evaluation at s, ξ is assumed unless a function is subscripted by $*$, and then we will assume it is evaluated at s_*, ξ_*.

Fourier Transform

We will briefly use two different Fourier transforms; one for functions $f : \mathbb{R}^n \to \mathbb{C}^m$,

$$\hat{f} : \mathbb{R}^n \to \mathbb{C}^m, \quad \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x}\,\mathrm{d}x,$$

(where the integral is interpreted as an m-vector of integrals) and one for functions $f : \mathbb{T}^n \to \mathbb{C}^m$,

$$\hat{f} : \mathbb{Z}^n \to \mathbb{C}^m, \quad \hat{f}(k) = \int_{\mathbb{T}^n} f(x)\mathrm{e}^{-2\pi\mathrm{i}k\cdot x}\,\mathrm{d}x.$$

These will be both written as $\hat{f}$ or $\mathcal{F}f$, and it will be clear from context which is being used.

A Fourier multiplier is a particular kind of operator on functions whose Fourier transform can be defined (first for a class of nice functions, and then by density to a larger space). Briefly, if $m = m(\xi)$ is some function of ξ, then we associate to it the multiplier

$$m(\partial)f := \mathcal{F}^{-1}(m(\xi)\mathcal{F}f),$$

and we say that the function m is the symbol of the multiplier. Not all choices of symbols m lead to a bounded operator $m(\partial)$; for further details, see for instance, [76].

Function Spaces

The spaces L^p, $p \in [1, \infty]$ are the usual Lebesgue spaces. C^s, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ denotes the space of $C^{\lfloor s \rfloor}$ functions whose $\lfloor s \rfloor$th derivative is $(s - \lfloor s \rfloor)$-Hölder. A subscript of 'loc' means that the function belongs to a 'local version' of the space (i.e. when restricted to a compact set). When the space is subscripted with a variable, it means that the condition defining the space is with respect to that variable. For instance,

$$f(x, y) \in L^1_x(\mathbb{R}) \iff \text{for almost every } y, \int_{\mathbb{R}} |f(x, y)| \, \mathrm{d}x < \infty.$$

For functions $u = u(x, t)$ of both space and time, we say $u \in L^p(0, T; X)$ if it belongs to the L^p Bochner space of X-valued functions. (See [69] for details.) The $L^2(0, T; L^2(Y))$ space can be safely identified with the more usual Lebesgue space $L^2([0, T] \times Y)$.

Asymptotic Notation

For any two functions $f, g : X \to (Y, \|\cdot\|)$, and an open set U containing a point x_0, we write:

1. $f = O(g)$ as $x \to x_0$, if there exists C such that for $x \in U$ sufficiently close to x_0,
$$\|f(x)\| \leq C\|g(x)\|.$$
2. $f = o(g)$ as $x \to x_0$, if
$$\frac{\|f(x)\|}{\|g(x)\|} \xrightarrow[x \to x_0]{} 0.$$

In addition, we write $A \lesssim B$ to mean that $A \leq CB$ for a positive constant C. If we want to stress the dependence of C on some parameters $a_1, \ldots, a_n$, then we write

$$A \lesssim_{a_1, \ldots, a_n} B.$$

1.4 Planar curves

Here, we introduce the basic theory of planar curves, and explain the particular 'uniform speed' parameterisations that we will use throughout this thesis.

Since we only cover the bare minimum required to understand our results, the reader who would like more details and background on the geometry of curves should consult some of the following text books: [71], [70], [21], [35], [75], [51].

Definition 1.6 (Planar curves, simple, closed)**.** A (planar) curve is a continuous map $z : [a, b] \to \mathbb{R}^2$. By an abuse of notation, we also refer to the image $z([a, b]) \subset \mathbb{R}^2$ as the curve z. If $z(t_1) \neq z(t_2)$ for all $t_1, t_2 \in [a, b)$ with $t_1 \neq t_2$, we say that the curve is simple. We say that a curve is closed if $z(a) = z(b)$, in which case we can by a slight abuse of notation, identify it with the periodic function $z : \mathbb{R}/((b-a)\mathbb{Z}) \to \mathbb{R}^2$. If the curve is k-times continuously differentiable, we say that it is a C^k curve.

Definition 1.7 (Parameterisations)**.** For a particular curve $z : [a, b] \to \mathbb{R}^2$, if the values $z(t)$ are specified by the parameter $t \in [a, b]$, a reparameterisation of z is an invertible map $\phi : [a, b] \to [c, d]$ such that $\tilde{z} : [c, d] \to \mathbb{R}^2$ defined by $\tilde{z}(r) = z(\phi^{-1}(r))$ is a curve with the same image $z([a, b]) = \tilde{z}([c, d])$. A parameterisation of a curve is a particular choice of the parameter space $t \in [a, b]$ and map z with the same image $C = z([a, b])$.

Definition 1.8 (Regular points and curves)**.** For the C^1 curve $z : [a, b] \to \mathbb{R}^2$, we say $z(c)$ is a regular point if $z'(c) \neq 0$. If every point in the curve is regular, we say that the curve is regular.

Proposition 1.9. *Any reparameterisation of a regular curve is regular.*

Proof. This follows from the chain rule, since if ϕ is a reparameterisation $\tilde{z} = z \circ \phi^{-1}$ then $\tilde{z}'(\phi(t)) \cdot \phi'(t) = z'(t) \neq 0$. The fact that $\phi' \neq 0$ (since it is a diffeomorphism) implies that $\tilde{z}'(\phi(t)) \neq 0$ at every t. $\square$

Definition 1.10 (Length)**.** The length of a C^1 curve $z : [a, b] \to \mathbb{R}^2$ is the positive quantity $L = \int_a^b |z'(t)| \, \mathrm{d}t$.

Now, we give the definition of an arc-length parameterisation.

Definition 1.11 (Arc-length parameterisation)**.** For a curve $z : [a, b] \to \mathbb{R}^2$, its arc-length reparameterisation $s : [a, b] \to [0, L]$ is given by the formula

$$s(t) = \int_a^t |z'(t')| \, \mathrm{d}t'.$$

The value $s(t)$ is the length of the part of the curve parameterised by the segment $[a,t]$. The resulting curve $\tilde{z} = z \circ s^{-1}$ is said to be parameterised by arc-length.

Proposition 1.12. *The arc-length parameterisation for a regular curve is a reparameterisation.*

Proof. This is because s is an increasing function with derivative $|z'|$ bounded away from 0. $\square$

For a vector $v = \binom{a}{b}$, we write $v^\perp = \binom{-b}{a}$ for its rotation by 90° anti-clockwise.

Definition 1.13 (Tangents and normals)**.** Suppose z is a curve parameterised by arc-length. Then a tangent to z at $z(s)$ is a vector $T \in \mathbb{R}^2$ such that $z'(s) \cdot T^\perp = 0$. A normal to $z(s)$ is a vector $N \in \mathbb{R}^2$ such that $z'(s) \cdot N = 0$. z' is a tangent vector, and $(z')^\perp$ is a normal vector.

If z is parameterised by arc-length, then $|z'(s)| = 1$, so z' is a unit tangent vector and $(z')^\perp$ is a unit normal.

The Jordan Curve Theorem asserts that a simple closed curve in $\mathbb{R}^2$ separates the plane into two disjoint components, which we will call the inside and outside of the curve. (For piecewise C^1 curves, this theorem is not too hard to prove, see for instance [63] for an elementary proof.) The inside of the curve is a bounded open set, and the outside is an unbounded open set. A normal N to z points inward if for sufficiently small $\epsilon > 0$, $z + \epsilon N$ is inside of the curve, and outward if it is outside of the curve.

Definition 1.14 (Orientation)**.** A C^1 curve z is said to be parameterised in the counter-clockwise sense, or positively oriented, if the normal vector $(z')^\perp$ points inward, otherwise it is said to be parameterised in the clockwise sense.

Proposition 1.15 (Frenet Formula)**.** *For a C^1 curve z parameterised with arc-length, the derivative of the tangent vector $T = z'$ is proportional to the normal $N = (z')^\perp$. This proportionality constant is called the curvature $\kappa \in \mathbb{R}$, i.e.*

$$T' = \kappa N,$$

and as a consequence,

$$N' = -\kappa T.$$

Proof. This follows from the fact that T is unit length at every point, so that $T' \cdot T = 2(|T|^2)' = 0$. Thus $T' = (T' \cdot T)T + (T' \cdot N)N = (T' \cdot N)N$. This gives the formula for the curvature $\kappa = T' \cdot N$. To conclude the evolution equation for the normal, simply notice that $N^\perp = (T^\perp)^\perp = -T$. $\square$

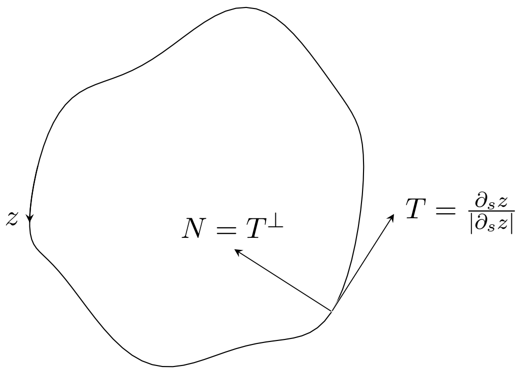

Figure 1.3: A positively oriented closed curve $z = z(s)$ with tangent $T = \partial_s z/|\partial_s z|$ and inward normal $N = T^{\perp}$.

Now, we define the parameterisation of curves that we will use throughout this thesis. The justification for this choice instead of the geometrically natural arc-length parameterisation is that we will be considering curves $z = z(t)$ that evolve in time, whose length $L = L(t)$ may also evolve in time. The analysis is made simpler if these curves are defined on a fixed parameter space at all times, which matches the formulation of the existence theorems.

Definition 1.16 (Uniform speed parameterisation)**.** The uniform speed parameterisation of a C^1 curve is the reparameterisation s/L, where s is the arc-length, and L is the total length of the curve.

Here, we list the consequences of this choice of parameterisation.

- All closed curves in uniform speed parameterisation are now defined on the common domain $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- z' is no longer unit length, but its length is the same at every point of the curve. So the unit tangent is $T = \frac{z'}{L}$, and the corresponding normal is $N = T^{\perp} = \frac{(z')^{\perp}}{L}$.
- The Frenet formulas take the form

$$\begin{cases} T' = L\kappa N, \\ N' = -L\kappa T. \end{cases}$$

- The curvature has the explicit formula $\kappa = T' \cdot N = L^{-3} z'' \cdot (z')^{\perp}$.

Chapter 2

Sharp fronts to Singular SQG equation

In this chapter, we derive the contour dynamics equation (CDE) solved by a sharp front solution to our equation (1.1).

2.1 Derivation of the contour dynamics equation

We will derive here the CDE for a sharp front of the following system:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ u = (\nabla^{\perp}K)*\theta. \end{cases} \tag{2.1}$$

The kernel $K \in C^{\infty}(\mathbb{R}^2 \setminus \{0\})$ can be any kernel that justifies our use of Divergence Theorem. In particular, we can use the kernels $K = \Lambda^{-\alpha}$, $\alpha \in (0,1)$ used in the equations (1.1) that are the main focus of this thesis.

Definition 2.1. We say that $\theta = \theta(x,t)$ for $x \in \mathbb{R}^2, t \geq 0$ is a weak solution to (2.1) if there exists $T > 0$ such that $\theta \in L^2(0,T;L^2(\mathbb{R}^2))$, $u = \nabla^{\perp}K * \theta \in L^2(0,T;L^2(\mathbb{R}^2;\mathbb{R}^2))$, and for any $\phi \in C_c^{\infty}((0,T)\times\mathbb{R}^2)$,

$$\int_0^T \int_{\mathbb{R}^2} \theta\partial_t\phi + u\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.2}$$

Definition 2.2 (Sharp front)**.** We say that a weak solution θ to (2.1) is a sharp front solution to (2.1) if

1. for each $t \in [0,T]$, there exists a bounded simply connected closed set $A(t)$ with C^2 boundary, and a function $z = z(s,t)$ in $C^2(\mathbb{T}\times[0,T])$ such that $z(\cdot,t)$

defines an anti-clockwise parameterisation of $\partial A(t)$,

2. θ is an indicator function for each $t \in (0, T)$,

$$\theta(x,t) = \mathbf{1}_{x\in A(t)} := \begin{cases} 1 & x \in A(t), \\ 0 & x \notin A(t). \end{cases}$$

Remark 2.3. It will be convenient (when we define the spine curve in Section 6.2) to consider the equivalent formulation $\theta := a\mathbf{1}_{x\in A(t)} + b$ for some fixed numbers $a, b \in \mathbb{R}$. In this case it is $\theta - b$ that is in $L^2(0, T; L^2)$. This serves to simplify calculations and make cancellations more apparent.

It will be useful to have coordinates defined on a fixed domain $\mathbb{T}$. Therefore, we will not be using arc-length coordinates, since the length of the curve may not be preserved. Instead, we will use the uniform speed parameterisation (see Definition 1.16, and also see Section 5.1.1).

Proposition 2.4 (Evolution of a sharp front)**.** *If $\theta = \mathbf{1}_A$ is a sharp front solution to (2.1), then the uniform speed counter-clockwise parameterisation $z : \mathbb{T} \to \mathbb{R}^2$ of ∂A with normal $N = \partial_s z^\perp$ satisfies the following CDE,*

$$\partial_t z \cdot N = \left(-\int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z)\,\mathrm{d}s_* \right) \cdot N =: -\mathcal{I}(z) \cdot N. \tag{2.3}$$

Proof. We deal with the two terms in (2.2) separately; Let $\nu = (\nu^1, \nu^2, \nu^3)$ be the outward unit normal to the surface $S = \{(x,t) : x \in \partial A(t)\} \subset \mathbb{R}^3$, with surface measure $\mathrm{d}\sigma(s)$. Consider the first term; integration by parts yields

$$\begin{aligned}
\int_{\mathbb{R}^2\times(0,T)} \theta \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t &= \int_{(x,t)\colon x\in A(t)} \nabla_{x^1,x^2,t} \cdot \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix} \mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}^2\times(0,T)} \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix} \cdot \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix} \mathrm{d}x\,\mathrm{d}t \\
&= \int_S \phi \nu^3 \,\mathrm{d}\sigma(x)\,\mathrm{d}t \\
&= \int_{s\in\mathbb{T}} \int_{t\geq 0} \phi(z,t) \partial_s z^\perp \cdot \partial_t z \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{t\geq 0} \int_{\partial A(t)} \phi(z,t) \partial_t z \cdot N \,\mathrm{d}l\,\mathrm{d}t.
\end{aligned}$$

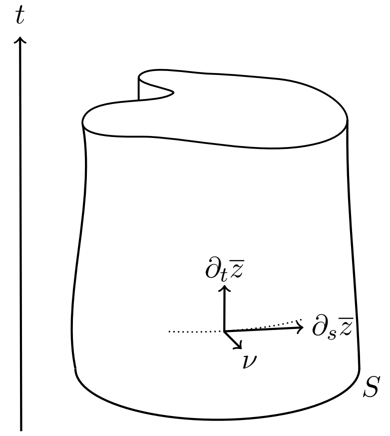

Figure 2.1: Setting $\overline{z}(s,t) := (z^1(s,t), z^2(s,t), t) \in \mathbb{R}^3$, this diagram depicts the outward normal $\nu = \partial_s\overline{z} \times \partial_t\overline{z}$ of the surface $S = \{(x,t) : x \in \partial A\}$, used in the application of the Divegence Theorem in dimension 3.

In the above, $\mathrm{d}l = |\partial_s z|\,\mathrm{d}s = L(t)\,\mathrm{d}s$ is a multiple of the arc-length measure on ∂A, and we have used the compact support away from $t = 0$ of ϕ when applying the Divergence Theorem (in dimension 3). Also, the explicit parameterisation in s, t was used to obtain the third component ν^3 of the outward normal ν (see Figure 2.1),

$$\nu = \begin{pmatrix} \partial_s z^1 \\ \partial_s z^2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \partial_t z^1 \\ \partial_t z^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \partial_s z^2 \\ -\partial_s z^1 \\ \partial_s z^\perp \cdot \partial_t z \end{pmatrix}.$$

Here only the indicator form of θ and the regularity of A are used. For the other term of (2.2), the properties of u are important. As u is the distributional perpendicular gradient of the convolution $K * \mathbf{1}_A$, it is given by a function away from ∂A.

We omit all appearances of t and consider only the spatial integral $\int_{\mathbb{R}^2}(\dots)\,\mathrm{d}x$ first. For $\delta \ll 1$, define the δ-neighbourhood of A,

$$A_\delta := \left\{x \in \mathbb{R}^2 : d(x, A) < \delta\right\} \supset A,$$

with $z_\delta := z - \delta N \notin A$ an explicit parameterisation of ∂A_δ using the same parameter s as z. Also define $\mathrm{d}l_\delta$ as the corresponding measure on ∂A_δ that uses the above parameter s, and write $T_\delta := \partial_s z_\delta$, $N_\delta := \partial_s z_\delta^\perp$. (For sufficiently small δ and smooth boundaries ∂A, $T(s) = T_\delta(s)$ and $N_\delta(s) = N(s)$.) We can rewrite the following as a limit $\delta \to 0$ (note that $-N_\delta$ is the outward normal to $A_\delta \subset \mathbb{R}^2$),

$$\begin{aligned}\int_A u \cdot (\nabla\phi)\theta \,\mathrm{d}x &= \int_A u \cdot \nabla\phi \,\mathrm{d}x \\ &= \lim_{\delta\downarrow 0} \int_{A_\delta} u \cdot \nabla\phi \,\mathrm{d}x \\ &= \lim_{\delta\downarrow 0} \int_{A_\delta} \nabla \cdot (u\phi) \,\mathrm{d}x \\ &= -\lim_{\delta\downarrow 0} \int_{\partial A_\delta} \phi(z_\delta) u(z_\delta) \cdot N_\delta \,\mathrm{d}l_\delta,\end{aligned}$$

where the third line follows due to the equality $\nabla \cdot \nabla^\perp = 0$, so that $\nabla \cdot u = 0$. Now, the integrand for fixed $\delta > 0$ is bounded since $z_\delta \notin A$. Indeed, expanding the definition of $u(z_\delta)$,

$$\begin{aligned}u(z_\delta) \cdot N_\delta &= \int_{y\in\mathbb{R}^2} 1_A(y) \nabla_x^\perp K(z_\delta - y) \cdot N_\delta \,\mathrm{d}y \\ &= \int_{y\in A} \nabla_x^\perp K(z_\delta - y) \cdot N_\delta \,\mathrm{d}y \\ &= \int_{y\in A} -\nabla_y^\perp K(z_\delta - y) \cdot N_\delta \,\mathrm{d}y \\ &= -\int_{y\in A} \nabla_y K(z_\delta - y) \cdot T_\delta \,\mathrm{d}y \\ &= -\int_{y\in A} \nabla_y \cdot (K(z_\delta - y) T_\delta) \,\mathrm{d}y,\end{aligned}$$

where we have used the fact that $N_\delta = T_\delta^\perp$ and for vectors v independent of y, $\nabla_y f \cdot v = \nabla_y \cdot (fv)$. An application of the Divergence Theorem then yields,

$$\begin{aligned}u(z_\delta) \cdot N_\delta &= \int_{\partial A} K(z_\delta - z_*) T_\delta \cdot N(s_*) \,\mathrm{d}l(s_*) \\ &= -\int_{\partial A} K(z_\delta - z_*) T(s_*) \cdot N_\delta \,\mathrm{d}l(s_*) \\ &= -\int_{\partial A} K(z_\delta - z_*)(T(s_*) - T_\delta) \,\mathrm{d}l(s_*) \cdot N_\delta.\end{aligned}$$

We have thus expressed the second term in (2.2) as:

$$\begin{aligned}\int_{\mathbb{R}^2} u \cdot \nabla\phi\theta \,\mathrm{d}x &= -\lim_{\delta\downarrow 0} \int_{\partial A_\delta} \phi(z_\delta) \left(-\int_{\partial A} K(z_\delta - z_*)(T_* - T_\delta) \,\mathrm{d}l(s_*) \cdot N_\delta \right) \mathrm{d}l_\delta \\ &= \int_{\partial A} \phi(z) \left(\int_{\partial A} K(z - z_*)(T_* - T) \,\mathrm{d}l_* \right) \cdot N \,\mathrm{d}l\end{aligned}$$

$$= \int_{\partial A} \phi(z) \left(\int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z) \, \mathrm{d}s_* \right) \cdot N \, \mathrm{d}l.$$

In the above lines the limit $\delta \downarrow 0$ was removed using the extra cancellation in the normal direction, $T_* \cdot N = (T_* - T) \cdot N$ (since $T|\partial_s z| = \partial_s z$). Hence, invoking the fundamental lemma of the calculus of variations by the arbitrariness of ϕ,

$$\partial_t z \cdot N = \left(- \int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z) \, \mathrm{d}s_* \right) \cdot N,$$

which is what we wanted. □

Remark 2.5 (Alternate forms of the CDE). Since the evoltuion is naturally only constrained in the normal direction, we can also write

$$\partial_t z = - \int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z) \, \mathrm{d}s_* + \lambda \partial_s z = \mathcal{I}(z) + \lambda \partial_s z,$$

for $\mathcal{I}(z)$ given by the integral term (2.3), and some $\lambda = \lambda(s, t)$ depending on the choice of parameterisation. (Any multiple of $\partial_s z$ can be reparameterised away without affecting the shape of $z(\cdot, t)$.) The evolution is written in this form in the papers [32] and [18] by Gancedo. We will also use this formulation in Section 4 of this thesis, where we prove the existence of sharp fronts in the analytic class.

Chapter 3

The Abstract Cauchy–Kowalevskaya Theorem

In this chapter, we state and prove the version of the abstract Cauchy–Kowalevskaya Theorem that we will use in Chapter 4.

3.1 The classical Cauchy–Kowalevskaya Theorem

The classical Cauchy–Kowalevskaya theorem is one of the few existence and uniqueness results that apply to a large class of PDEs. It provides local existence and uniqueness for solutions $u = u(x,t) \in \mathbb{R}^n$, $(t \geq 0, x \in U \subset \mathbb{R}^m)$ to nonlinear systems

$$\begin{cases} \partial_t u(x,t) + E(x,t,u(x,t),u_x(x,t)) = 0, & t \geq 0, x \in U, \\ u|_{t=0} = 0, & x \in U. \end{cases} \tag{3.1}$$

So long as the nonlinearity E and the initial data are analytic, a unique analytic solution is guaranteed to exist for short times. (One can phrase this slightly more generally with a fully nonlinear equation $G(x,u,u_x) = 0$ and more general initial conditions using characteristic surfaces, but the two turn out to be equivalent: see the presentation in [31].) More precisely,

Definition 3.1 (Analytic functions)**.** A function $u : \mathbb{R}^n \to \mathbb{R}$ is (real) analytic near x_0 if it has the power series expansion valid on a neighbourhood of x_0,

$$u(x) = \sum_{\alpha \in \mathbb{N}_0^n} u_\alpha (x - x_0)^\alpha.$$

(Here, α runs through all multi-indices in $\mathbb{N}_0^n$, as defined in Chapter 1.) If it is analytic near every x_0 in Ω, we say u is analytic in Ω and write $u \in C^\omega(\Omega)$. A function $u : \mathbb{R}^n \to \mathbb{R}^m$, $u = (u_1, \ldots, u_m)^T$ is said to be real analytic if each component function u_i is real analytic.

More details about real or complex analytic functions of one or several variables can for instance be found in the books [72], [44], and [9].

Theorem 3.2 (Cauchy–Kowalevskaya)**.** *Let $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$ be an analytic function. Then there exists $\epsilon > 0$ such that the Cauchy problem (3.1) has a unique solution $u = u(x,t)$ that is in $C^\omega(\mathbb{R}^n \times (-\epsilon, \epsilon))$.*

A particular case of this theorem was proven by Cauchy in [12], and the full version was proven by Kowalevskaya in [80]. The standard proof is available in many textbooks like [24] and [31], and proceeds by the method of majorants: in a few words, this proof proceeds by computing a formal power series for the solution and verifying that the power series converges on a small enough ball by comparison with coefficients of a known explicit power series.

A different proof by Nagumo [52] using the Schauder fixed point theorem relies on the following lemma (as explained in the introduction of the paper [81] of Wolfgang, which also gives another proof using the Contraction Mapping Theorem)

Lemma 3.3 (Nagumo)**.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, $f : \Omega \to \mathbb{C}$ be holomorphic and $s > 0$. Then*

$$|f(z)| \lesssim \frac{1}{d(z, \partial\Omega)^s} \implies |f_{z_j}(z)| \lesssim (s+1)\frac{1}{d(z,\partial\Omega)^{s+1}}.$$

This bound follows easily from Cauchy's integral representation in one dimension, $f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)}{w-z}\,\mathrm{d}w$. It turns out that using inequalities of a similar type (which we will call Cauchy-type inequalities) allows one to generalise the Cauchy–Kowalevskaya Theorem, as we explain in the next section.

3.2 Generalisation to scales of Banach spaces

The classical Cauchy–Kowalevskaya theorem is remarkable and important because there exist PDEs with smooth coefficients and no solution, as in the example of Lewy [45]. However, there are many PDEs of interest for which we do not expect solutions to be analytic in all variables like the heat equation, or more generally less well behaved equations with an operator like the heat operator (see for instance the paper [74]). Therefore, a number of generalisations of the Cauchy–Kowalevskaya theorem

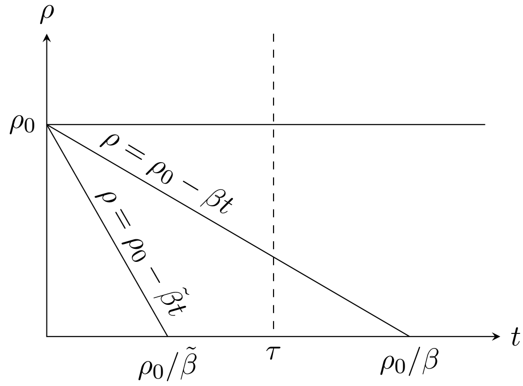

Figure 3.1: An illustration of the various parameters in the definitions of the spaces X_ρ, $X_{\rho,\tau}$, and $Y_{\rho,\beta,\tau}$. In particular, note that if $\tau \leq \rho_0/\beta$, then $Y_{\rho_0,\beta,\tau} \subset X_{\rho_0,\tau}$: the radius of analyticity for functions in $Y_{\rho_0,\beta,\tau}$ is allowed to shrink at the rate β at worst. Also, if $\beta < \tilde{\beta}$ and $\tau \leq \rho_0/\tilde{\beta}$, then $Y_{\rho,\beta,\tau} \subset Y_{\rho,\tilde{\beta},\tau}$.

have been proven over the years that can prove local existence and uniqueness in a class of functions that are only continuous (or C^1) in time and analytic in space.

Generalisations where the analyticity is measured with a scale of Banach spaces, building on the above mentioned work of Nagumo [52] were first considered by Ovsjannikov in [61], [60] (and more recently [62]), Yamanaka in [83] (and related papers [82], [84], [39], [40], [85], and [86]). Various refinements and variations have been made by a large list of authors, such as Nirenberg, Nishida, DuChateau, Asano, Safonov, Sammartino, Caflisch, Tutschke, and Koike, in the papers [55], [43], [53], [22], [23], [56], [2], [1], [73], [7] [79], [3], and [78]. The version which we present below follows the notation of the paper [74] by Sammartino and Caflisch, which is a variant of the main theorem of the paper [73] by Safonov. However, we could not produce a proof of the exact statement of the theorem stated in [74]. Instead, we have changed assumption (CK3) below so that the Cauchy-type estimate remains valid for $\beta > \beta_0$. The version here is sufficient for our purposes (and many others) as operators of order at most one will satisfy our version of (CK3).

Because of this difference, we present a full proof. The strategy of the proof is based on the methods of [48] and [2]. After the proof, we also make some remarks about the differences in the proof from [48].

We begin with some definitions. The parameter $\rho > 0$ in the classical setting is the radius of analyticity, or the thickness of the strip of analyticity; see Figure 3.1 for a graphical illustration of how the parameters relate to the spaces. (In particular, it is natural to ask for the nesting property of Banach scales below.)

Definition 3.4. Let $\rho_0 > 0$. A Banach scale $\{X_\rho, 0 < \rho < \rho_0\}$ with norms $\|\cdot\|_\rho$ is a collection of Banach spaces such that $X_{\rho'} \subset X_{\rho''}$ with $\|\cdot\|_{\rho''} \leq \|\cdot\|_{\rho'}$ whenever $\rho'' \leq \rho' \leq \rho_0$.

Definition 3.5. Given a Banach scale X_ρ, $\tau > 0$ and $0 < \rho \leq \rho_0$ and $R > 0$,

1. $X_{\rho,\tau}$ is the set of all functions $u(t)$ from $[0,\tau]$ to X_ρ endowed with the norm

$$\|u\|_{\rho,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_\rho.$$

2. $Y_{\rho,\beta,\tau}$ is the set of functions u such that for $t \in [0,\tau]$, $u(t) \in X_{\rho-\beta t}$, with the norm

$$\|u(t)\|_{\rho,\beta,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_{\rho-\beta t}.$$

3. We will denote by $X_{\rho,\tau}(R)$ and $Y_{\rho,\beta,\tau}(R)$ the balls of radius R in $X_{\rho,\tau}$ and $Y_{\rho,\beta,\tau}$ respectively.

Theorem 3.6 (Abstract Cauchy–Kowalevskaya Theorem)**.** *Suppose that there exist $\rho_0 > 0$, $R > 0, \beta_0 > 0$, and $0 < T < \rho_0/\beta_0$, such that the following assumptions hold:*

(CK1) *For every pair ρ, ρ' such that $0 < \rho' < \rho < \rho_0 - \beta_0 T$ and every $u \in X_{\rho,T}(R)$, the function*[1] *$F(t,u) : [0,T) \to X_{\rho'}$ is continuous.*

(CK2) *For every ρ such that $0 < \rho \leq \rho_0 - \beta_0 T$, the function $F(t,0) : [0,T) \to X_{\rho,T}(R)$ is continuous in t, and*

$$\|F(t,0)\|_{\rho_0-\beta_0 t} \leq R_0 < R.$$

(CK3) *For any numbers $\beta \geq \beta_0$, $s < t < \min(T, \rho_0/\beta)$, $\rho' > 0$, any (continuous) function $\rho(s)$ such that $0 < \rho' < \rho(s) < \rho_0 - \beta s$, and any $u_1, u_2 \in Y_{\rho_0,\beta,T^*}(R)$, we have for a constant C independent of β,*

$$\|F(t,u_1) - F(t,u_2)\|_{\rho'} \leq C \int_0^t \frac{\|u_1(s) - u_2(s)\|_{\rho(s)}}{\rho(s) - \rho'} \, \mathrm{d}s.$$

Then there exist $\beta > \beta_0$, and $T^* \leq T$ such that there is a unique u belonging to $Y_{\rho_0,\beta,T^*}(R)$ that solves the equation

$$u = F(t,u).$$

[1]By $F(t,u)$, we mean the function F has functional dependence on u, which may include operators applied to u like ∇u.

Proof. Let β_0, ρ_0, T, R_0, and R be fixed constants as in the theorem statement. We introduce the following weighted Banach space for $\gamma \in (0,1)$ arbitrary but fixed, $\beta \gg 1$ to be chosen later, and $T^* := T^*(\beta) := \min(T, \rho_0/\beta) \leq T$,

$$\mathbb{S}^{\gamma,\beta} = \{u : [0, T^*) \to X_{\rho_0 - \beta T^*} : \|u\|^{(\gamma,\beta)} < \infty\},$$

where the weighted norm $\|u\|^{(\gamma,\beta)}$ is defined by

$$\|u\|^{(\gamma,\beta)} = \sup_{\substack{t<T^* \\ 0<\rho'<\rho_0-\beta t}} \left(1 - \frac{\beta t}{\rho_0 - \rho'}\right)^{\gamma} \|u(t)\|_{\rho'}. \tag{3.2}$$

Note that $\|u\|^{(\gamma,\beta)} \leq \|u\|_{\rho_0,\beta,T^*}$. If $0 < \tilde{\beta} < \beta$, then $\rho_0 - \beta t < \rho_0 - \tilde{\beta} t$, so making the choice $\rho' = \rho_0 - \beta t$, we have

$$\left(1 - \frac{\tilde{\beta} t}{\rho_0 - \rho'}\right)^{\gamma} = \left(\frac{\beta - \tilde{\beta}}{\beta}\right)^{\gamma}.$$

This implies the following inequalities for $0 < \tilde{\beta} < \beta$,

$$\|u\|^{(\gamma,\beta)} \leq \|u\|_{\rho_0,\beta,T^*} \leq \left(\frac{\beta}{\beta - \tilde{\beta}}\right)^{\gamma} \|u\|^{(\gamma,\tilde{\beta})}. \tag{3.3}$$

We first quickly verify that $\mathbb{S}^{\gamma,\beta}$ is Banach: Suppose u_n is Cauchy in the $\mathbb{S}^{\gamma,\beta}$ norm. Then for each $t < T^*, \rho' < \rho - \beta t$,

$$\|u_n(t) - u_m(t)\|_{\rho'} \leq \frac{\|u_n - u_m\|^{(\gamma,\beta)}}{(\rho_0 - \rho' - \beta t)^{\gamma}}, \tag{3.4}$$

so $u_n(t)$ is Cauchy in the Banach space $X_{\rho'}$, allowing us to define $u(t) := \lim_{n\to\infty} u_n(t)$ as an element of $X_{\rho'}$. As $\sup_n \|u_n\|^{(\gamma,\beta)} < C_0$ for some $C_0 > 0$, from (3.4) we deduce the bound

$$\|u(t) - u_n(t)\|_{\rho'} \leq \frac{2C_0}{(\rho_0 - \rho' - \beta t)^{\gamma}}. \tag{3.5}$$

This gives $\|u(t)\|_{\rho'} \leq \frac{3C_0}{(\rho_0 - \rho' - \beta t)^{\gamma}}$; multiplying by the denominator of the right-hand side, and taking a supremum over all allowable t and ρ' proves that $u \in \mathbb{S}^{(\gamma,\beta)}$. Then, (3.5) shows that $u_n \to u$ in $\mathbb{S}^{(\gamma,\beta)}$.

Contraction-type inequality

Here, we prove that for any $\beta \geq \beta_0$, $\gamma \in (0,1)$, and $u, v \in Y_{\rho_0,\beta,T^*}(R)$, we have[2]

$$\|F(t,u) - F(t,v)\|^{(\gamma,\beta)} \leq \frac{C 2^{1+\gamma} \|u-v\|^{(\gamma,\beta)}}{\gamma\beta}. \tag{3.6}$$

In particular, if

$$\beta > \frac{C 2^{1+\gamma}}{\gamma}, \tag{3.7}$$

then F is a contraction. Define for $0 < \rho' < \rho_0 - \beta s$, $s < T^*$,

$$\rho(s) := \frac{\rho' + \rho_0 - \beta s}{2}.$$

As an average of ρ' and $\rho_0 - \beta s$, we have $\rho' < \rho(s) < \rho_0 - \beta s$. So we can apply (CK3). If we define $\lambda(s)$ by $\rho(s) =: \rho' + \frac{\lambda(s)}{2}$, i.e.

$$\lambda(s) := \rho_0 - \rho' - \beta s, \tag{3.8}$$

then

$$\rho(s) - \rho' = \frac{\lambda(s)}{2} = \rho_0 - \rho(s) - \beta s. \tag{3.9}$$

So from (CK3), we obtain for $t < T^*$,

$$\begin{aligned}
&\|F(t,u) - F(t,v)\|_{\rho'} \\
&\leq C \int_0^t \frac{\|u-v\|_{\rho(s)}}{\rho(s) - \rho'} \, \mathrm{d}s \\
&= C \int_0^t \frac{\|u-v\|_{\rho(s)}}{\rho(s) - \rho'} \cdot \underbrace{\frac{(\rho_0 - \rho(s) - \beta s)^\gamma}{(\rho_0 - \rho(s))^\gamma}}_{=\left(1 - \frac{\beta s}{\rho_0 - \rho(s)}\right)^\gamma} \frac{(\rho_0 - \rho(s))^\gamma}{(\rho_0 - \rho(s) - \beta s)^\gamma} \, \mathrm{d}s && (3.10) \\
&\leq C (\rho_0 - \rho')^\gamma \|u-v\|^{(\gamma,\beta)} \int_0^t \frac{\mathrm{d}s}{(\lambda(s)/2)^{1+\gamma}} && (3.11) \\
&= C (\rho_0 - \rho')^\gamma 2^{1+\gamma} \|u-v\|^{(\gamma,\beta)} \int_0^t \frac{\mathrm{d}s}{(\rho_0 - \rho' - \beta s)^{1+\gamma}} && (3.12) \\
&= \frac{C (\rho_0 - \rho')^\gamma 2^{1+\gamma} \|u-v\|^{(\gamma,\beta)}}{\gamma\beta} \left(\frac{1}{(\rho_0 - \rho' - \beta t)^\gamma} - \frac{1}{(\rho_0 - \rho')^\gamma} \right) && (3.13)
\end{aligned}$$

[2]Note that this implies $u, v \in \mathbb{S}^{\gamma,\beta}$, by (3.3).

$$
\begin{aligned}
&= \frac{C2^{1+\gamma}\|u-v\|^{(\gamma,\beta)}}{\gamma\beta}\left(\underbrace{\frac{(\rho_0-\rho')^\gamma}{(\rho_0-\rho'-\beta t)^\gamma}}_{=\left(1-\frac{\beta t}{\rho_0-\rho'}\right)^{-\gamma}>1}-1\right)\\
&\le \frac{C2^{1+\gamma}\|u-v\|^{(\gamma,\beta)}}{\gamma\beta}\cdot\frac{1}{\left(1-\frac{\beta t}{\rho_0-\rho'}\right)^\gamma}.
\end{aligned}
$$

In going from (3.10) to (3.11), we used the definition of $\|\cdot\|^{(\gamma,\beta)}$ in (3.2), $\rho(s)>\rho'$ and (3.9) (keeping in mind that $\rho_0,\rho(s),\rho'$ are not the same quantities), then (3.8) is used to obtain (3.12), and then the integral is directly computed to obtain (3.13).

Multiplying both sides by $\left(1-\frac{\beta t}{\rho_0-\rho'}\right)^\gamma$ and taking a supremum over t and ρ such that $t<T^*,\rho'<\rho_0-\beta t$ yields the desired inequality (3.6).

Iteration scheme

Set $u_0:=0$ and inductively define $u_n:=F(t,u_{n-1})$. Then (CK2) implies that

$$
\|u_1\|_{\rho_0,\beta,T^*}\le\|u_1\|_{\rho_0,\beta_0,T}\le R_0<R. \tag{3.14}
$$

The goal is to iteratively apply (3.6), as usual for contraction maps. For this, we need to show that $u_n\in Y_{\rho_0,\beta_0,T^*}(R)$ for every $n>1$. This will give a second condition depending on the difference $R-R_0>0$, requiring that β be large enough to satisfy it.

Control of Y_{ρ_0,β_0,T^*} norm of u_n

Define the auxiliary sequence b_k $(k\ge 1)$ by

$$
b_k=\beta\left(1-\frac{1}{2^k}\right).
$$

Note that b_k is an increasing sequence with $b_k\to\beta$. Thus, for every $k\ge 1$,

$$
b_k\in(\beta/2,\beta). \tag{3.15}
$$

Since we want to apply (3.6) (which is only valid for $\beta\ge\beta_0$) with b_k in place of β, our construction requires[3]

$$
\beta\ge 2\beta_0. \tag{3.16}
$$

[3]This is hardly optimal, since we can instead use $\tilde b_k=\beta(1-2^{-k-k_0})$ with $k_0\gg 1$ instead, but we will refrain from improving the bound on β in this way to avoid complicating the other estimates.

Also, note that $\left(\frac{\beta}{\beta-b_k}\right)^\gamma = 2^{\gamma k}$. Therefore, for $k \geq 1$, by choosing $u = u_{k+1} - u_k$ in the right inequality of (3.3) and applying (3.6) k times,

$$\begin{aligned}\|u_{k+1}-u_k\|_{\rho_0,\beta,T^*} &\leq 2^{\gamma k}\|u_{k+1}-u_k\|^{(\gamma,b_k)} \\ &\leq 2^{\gamma k}\left(\frac{C2^{1+\gamma}}{\gamma b_k}\right)^k \|u_1-u_0\|^{(\gamma,b_k)} \\ &\leq \left(\frac{C4^{1+\gamma}}{\gamma\beta}\right)^k \|u_1\|^{(\gamma,b_k)}, \end{aligned} \tag{3.17}$$

since $u_0 = 0$, and $b_k > \beta/2$ from (3.15). Then (3.14) and the left inequality of (3.3) implies

$$\|u_1\|^{(\gamma,b_k)} \leq \|u_1\|_{\rho_0,b_k,T^*} \leq R_0. \tag{3.18}$$

Applying (3.17) and (3.18) (and the fact that $u_0 = 0$), we have for $n \geq 2$,

$$\begin{aligned}\|u_n\|_{\rho_0,\beta,T^*} &\leq \|u_0\|_{\rho_0,\beta,T^*} + \|u_1-u_0\|_{\rho_0,\beta,T^*} + \sum_{k=1}^{n-1}\|u_{k+1}-u_k\|_{\rho_0,\beta,T^*} \\ &\leq R_0 + \sum_{k=1}^{n-1}\left(\frac{C4^{1+\gamma}}{\gamma\beta}\right)^k \|u_1\|^{(\gamma,b_k)} \\ &\leq R_0\sum_{k=0}^{\infty}\left(\frac{C4^{1+\gamma}}{\gamma\beta}\right)^k \\ &= \frac{R_0\gamma\beta}{\gamma\beta - C4^{1+\gamma}}.\end{aligned}$$

Therefore, in order to ensure that $u_n \in Y_{\rho_0,\beta,T^*}(R)$, we need

$$\beta > \frac{C4^{\gamma+1}}{\gamma(R-R_0)}. \tag{3.19}$$

Existence and uniqueness of solution

Let β be so large that

$$\beta > \max\left(\frac{C2^{1+\gamma}}{\gamma}, 2\beta_0, \frac{C4^{\gamma+1}}{\gamma(R-R_0)}\right).$$

Then (3.7), (3.16) and (3.19) are satisfied. Thus for some $R_0 < R_1 < R$, we have $u_n \in \overline{Y_{\rho_0,\beta,T^*}(R_1)}$, and the Contraction Mapping Theorem with (3.6) implies that there is a unique solution to $u = F(t,u)$ in $Y_{\rho_0,\beta,T^*}(R)$. $\square$

3.2.1 Discussion of the proof

As mentioned earlier, our assumption (CK3) does not match the analogous assumption of [74]. Furthermore, while F indeed satisfies a contraction-type inequality in the weighted norm $\|u\|^{(\gamma)} := \sup_{\rho'<\rho_0-\beta t}(\rho_0 - \rho' - \beta t)^\gamma \|u(t)\|_{\rho'}$ for $\beta \gg 1$ (which is what is proven in [48]), it does not seem possible to control the Y_{ρ_0,β,T^*} norm of the successive iterates u_n. This is because the right inequality of (3.3) is not true for the norm $\|\cdot\|^{(\gamma)}$, since if one tries to similarly use $\rho' = \rho_0 - \beta t$ to bound $\|u\|_{\rho_0,\beta,T^*}$, one finds possible blow-up at $t = 0$. Notably, Safonov [73] uses a similar collection of spaces, and the norm $\|\cdot\|^{(\gamma)}$ above, but a slightly different collection of assumptions for a 'differential' version of the abstract Cauchy–Kowalevskaya theorem allow him to complete the proof.

Instead, with inspiration from Asano [2], we defined $\|\cdot\|^{(\gamma,\beta)}$ with the normalised weight $\frac{(\rho'-\rho_0-\beta t)^\gamma}{(\rho'-\rho_0)^\gamma} = \left(1 - \frac{\tilde{\beta} t}{\rho_0-\rho'}\right)^\gamma$. We also borrowed from Asano the idea of introducing an auxillary sequence b_k, so that the norms $\|u\|_{\rho_0,\beta,T^*}$ can be controlled using (3.3). (Presumably, this is what is meant by Safonov [73] in saying that in the result of Asano, 'the domain of existence shrinks at each step of iteration.')

Finally, we note that although the assumption (CK3) is sufficient for many applications, it is not optimal. From the proof, we see that we only need the assumption to hold for a certain sufficiently large $\beta \gg \beta_0$, and no larger.

3.3 Example application

We finish this chapter with an alternative proof of the classical Cauchy–Kowalevskaya theorem, following the proof that Safonov gives using his version of the abstract Cauchy–Kowalevskaya theorem in [73].

Proof of Theorem 3.2. Solving the equation (3.1) is equivalent to solving the time-integrated system,

$$x \in U \subset \mathbb{R}^m, \quad t \geq 0, \quad u(x,t) \in \mathbb{R}^n,$$
$$u(x,t) = F(t,u)(x,t) = \int_0^t E(x,s,u,u_x)\,\mathrm{d}s. \tag{3.20}$$

Indeed, if a continuous function u solves (3.20) and E is analytic in t, then u is also analytic, for instance by the (simpler) Cauchy–Kowalevskaya theorem for ordinary differential equations. However, the analyticity of F in t is not required for the abstract Cauchy–Kowalevskaya theorem to work.

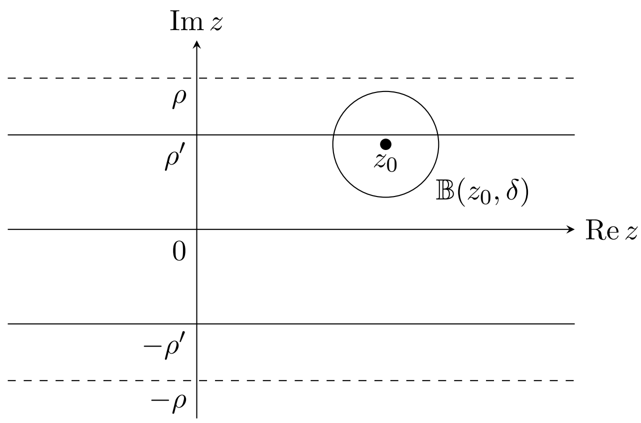

Figure 3.2: The application of Cauchy's theorem to derive the Cauchy estimate for analytic functions of one complex variable $z = s + \mathrm{i}\tilde{s}$. Here, ρ is the thickness of the strip of analyticity of $f = f(z)$, $z_0 \in \{|\operatorname{Im} z| < \rho'\}$ and $0 < \delta < \rho - \rho'$.

Define the Banach scale X_ρ as the family in $\rho > 0$ of Banach spaces of n-vectors of analytic functions u in m variables that are (real) analytic on U, such that they have an analytic extension (also written u) to the complex neighbourhood of U,

$$U_\rho := \{x + \mathrm{i}\tilde{x} \in \mathbb{C}^m : x \in U, |\tilde{x}| < \rho\},$$

where each $u : U_\rho \to \mathbb{C}^n$ satisfies

$$\|u\|_\rho := \sup_{z \in U_\rho} |u(z)| + \sup_{z \in U_\rho} |u_x(z)| < \infty.$$

If E is analytic in all variables, then in particular (CK1) and (CK2) are clearly satisfied for some $\rho_0, \beta_0, T, R_0, R$ that depend only on E. That is, F is continuous, and $F(t, 0), F_x(t, 0)$ take values in a bounded set for bounded values of x, t, respectively. So we only need to check the Cauchy estimate (CK3).

Suppose $0 < \rho' < \rho < \rho_0$. Then for any analytic function $f \in X_\rho$,

$$\|f\|_{\rho'} \leq \frac{C}{\rho - \rho'} \sup_{w \in U_\rho} |f(w)|, \tag{3.21}$$

for a constant C depending on ρ_0, n and m only. This is easy to see first in dimension $m = n = 1$: if $U \subset \mathbb{R}$, $U_\rho \subset \mathbb{C}$ and $f(z) \in \mathbb{C}$, then the one-dimensional Cauchy's theorem gives for $z \in U_{\rho'}$, and $\rho - \rho' > \delta > 0$ (see Figure 3.2),

$$|f'(z)| = \left| \frac{1}{2\pi \mathrm{i}} \int_{|w-z|=\delta} \frac{f(w)\,\mathrm{d}w}{(z-w)^2} \right| \leq C_0 \frac{\sup_{|w-z|=\delta} |f(w)|}{\rho - \rho'} \leq C_0 \frac{\sup_{w \in U_\rho} |f(w)|}{\rho - \rho'}.$$

C_0 is a fixed universal constant. Therefore,

$$\|f\|_{\rho'} \leq \left(1 + \frac{C_0}{\rho - \rho'}\right) \sup_{w \in U_\rho} |f(w)| \leq \frac{C_1}{\rho - \rho'} \sup_{w \in U_\rho} |f(w)|.$$

C_1 depends possibly on ρ_0. The same argument applied to the components of the gradient f_x in the general case $m, n \geq 1$ gives (3.21). Note that for any functions $u, v \in X_\rho$, by the mean value inequality, there is some constant C_2 depending on ρ_0 and the derivatives in the 3rd and 4th variables of E such that

$$|E(t,x,u,u_x) - E(t,x,v,v_x)| \leq C_2 \Big(|u(z,t) - v(z,t)| + |u_x(z,t) + v_x(z,t)| \Big).$$

Set $\beta \geq \beta_0, T^* = \min(T, \rho_0/\beta)$ as in the statement of (CK3). Allowing $\rho = \rho(s)$ to vary continuously in the interval $(\rho', \rho_0 - \beta T^*)$, we therefore have for $u, v \in Y_{\rho_0,\beta,T^*}$,

$$\begin{aligned}
&\|F(t,u) - F(t,v)\|_{\rho'} \\
&\leq \int_0^t \|E(s,x,u,u_x) - E(s,x,v,v_x)\|_{\rho'}\,\mathrm{d}s \\
&\leq C_2 \int_0^t \|u - v\|_{\rho'} + \|u_x - v_x\|_{\rho'}\,\mathrm{d}s \\
&\leq CC_2 \int_0^t \frac{\sup_{z \in U_{\rho(s)}} |u(z,t) - v(z,t)| + \sup_{z \in U_{\rho(s)}} |u_x(z,s) - v_x(z,s)|}{\rho(s) - \rho'}\,\mathrm{d}s \\
&= C_3 \int_0^t \frac{\|u - v\|_{\rho(s)}}{\rho(s) - \rho'}\,\mathrm{d}s,
\end{aligned}$$

which is the assumption (CK3), which means that Theorem 3.6 applies. We conclude that there is for short time, a spatially analytic solution to (3.20), and therefore an analytic solution to (3.1). □

Chapter 4

Existence of Analytic Sharp Fronts

In this chapter, we carry out the computations necessary to put the sharp front equation (4.43) in a form amenable to the abstract Cauchy–Kowalevskaya theorem of the previous chapter. To slightly simplify the computations, we will (in this chapter only) write

$$\beta := 1 + \alpha \in (1, 2),$$

and use $[0, 1)$ as a fundamental domain of $\mathbb{T}$. Thus, the generalised SQG equation (1.1) is

$$\theta_t + (|\nabla|^{-2+\beta}\nabla^{\perp}\theta) \cdot \nabla\theta = 0,$$

and the corresponding sharp front equation is

$$z_t(s,t) \cdot N(s,t) = -\int_{\mathbb{T}} \frac{(z_s(s_*,t) - z_s(s,t)) \cdot N(s,t)}{|z(s_*,t) - z(s,t)|^{\beta}} \, \mathrm{d}s_*.$$

By a suitable reparameterisation in the curve parameter $s \in \mathbb{T}$, this is equivalent to the contour dynamics equation where $s \in \mathbb{T}$, $t \geq 0$,

$$z_t(s,t) = -\int_{\mathbb{T}} \frac{z_s(s_*,t) - z_s(s,t)}{|z(s_*,t) - z(s,t)|^{\beta}} \, \mathrm{d}s_* + \lambda(s,t) z_s(s,t) \tag{4.1}$$

$$=: \zeta(s,t) + \lambda(s,t) z_s(s,t), \tag{4.2}$$

for some function $\lambda = \lambda(s,t) \in \mathbb{R}$, that is also periodic in $s \in \mathbb{T}$, which we may assume (by reparameterising if necessary) to additionally satisfy $\lambda(0,t) = 0$ for every $t \geq 0$. The integral term $\zeta(s,t)$ is defined by (4.1).

The corresponding CDE for the Euler equation with $\beta = 0$, termed the vortex patch problem, was first derived by Zabusky et. al. [87] and its systematic study can be found in the book [49]. The study of sharp fronts for the SQG equation was initiated in the thesis of Rodrigo [68], where existence and uniqueness was proven in the class of smooth periodic graphs. Gancedo proved existence in [32] and later uniqueness [18] for Sobolev data. The interpolated equations for $\beta \in (0,1)$ was further studied in [32], [33], and [42]. More recently the more singular sharp fronts for $\beta \in (1,2)$ as in this thesis, or other variants of SQG) have been studied in [14], [13], and [57].

4.1 Preliminaries

To obtain a useful expression for λ, we will further reparameterise s so that

$$|z_s(s,t)|^2 = z_s(s,t) \cdot z_s(s,t) = L(t)^2, \quad s \in \mathbb{T}, t \geq 0.$$

That is to say, $z_s \cdot z_{ss}$ identically vanishes. In this final parameterisation, s is not arc-length, but the vectors $z_s(s,t)$ have for each time t a length $L(t)$ that is independent of the parameter s, and is geometrically the total length of the curve z at time t, i.e. $L(t) = \int_{\mathbb{T}} |z_s(s,t)| \, \mathrm{d}s$ (as in Definition 1.16). As in [32], the function λ can then be written explicitly in terms of z as follows. Taking the s derivative of z_t in equation (4.2) we find the equation for z_{st},

$$z_{st} = \zeta_s + \lambda z_{ss} + \lambda_s z_s. \tag{4.3}$$

Therefore

$$\frac{1}{2}(L^2)' = L'L = z_s \cdot z_{st} = z_s \cdot \zeta_s + \lambda_s L^2, \tag{4.4}$$

where we have substituted (4.3) in the last equality. Now (4.4) yields

$$\frac{L'}{L} = \frac{1}{L^2} z_s \cdot \zeta_s + \lambda_s. \tag{4.5}$$

Since λ is a periodic function in s, integrating (4.5) over $\mathbb{T}$, we obtain

$$\frac{L'}{L} = \int_{\mathbb{T}} \frac{z_s}{L^2} \cdot \zeta_s \, \mathrm{d}s. \tag{4.6}$$

Hence, integrating (4.5) from 0 to s, using $\lambda(0,t) = 0$ and dividing by L^2, we see that (using (4.6))

$$\lambda(s,t) = s\frac{L'(t)}{L(t)} - \int_0^s \frac{z_s(s_1,t)}{L(t)^2}\cdot\zeta_s(s_1,t)\,\mathrm{d}s_1$$
$$= -s\int_{\mathbb{T}} \frac{z_s(s_1)}{|z_s(s_1)|^2}\cdot\partial_{s_1}\int_{\mathbb{T}}\frac{z_s(s_0)-z_s(s_1)}{|z(s_0)-z(s_1)|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1$$
$$+\int_0^s \frac{z_s(s_1)}{|z_s(s_1)|^2}\cdot\partial_{s_1}\int_{\mathbb{T}}\frac{z_s(s_0)-z_s(s_1)}{|z(s_0)-z(s_1)|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1, \tag{4.7}$$

(hiding the t dependence for legibility reasons) which is an expression for λ completely determined by the values of z and z_s. Conversely, if λ is defined by (4.7), and z evolves via (4.1), then $\lambda(0,t)=0$ for every t, and differentiating (4.1) readily yields

$$\frac{1}{2}\partial_t|z_s|^2 = \zeta_s\cdot z_s + \frac{\lambda}{2}\partial_s|z_s|^2 + \lambda_s|z_s|^2 = \frac{\lambda}{2}\partial_s|z_s|^2 + \mu(t)|z_s|^2,$$

where $\mu(t)$ is the following function of t only,

$$\mu(t) = \lambda_s(s,t) + \frac{\zeta_s(s,t)\cdot z_s(s,t)}{|z_s(s,t)|^2}$$
$$= -\int_{\mathbb{T}} \frac{z_s(s_1)}{|z_s(s_1)|^2}\cdot\partial_{s_1}\int_{\mathbb{T}}\frac{z_s(s_0)-z_s(s_1)}{|z(s_0)-z(s_1)|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1.$$

One obtains by the method of characteristics for $F(s,t) = |z_s|^2(s,t)$, $F_0(s) = |z_s|^2(s,0)$,

$$F(s,t) = F_0\left(s+\int_0^t \lambda(s,\tau)\,\mathrm{d}\tau\right)\exp\left(2\int_0^t \mu(\tau)\,\mathrm{d}\tau\right).$$

Hence, if $\frac{\mathrm{d}F_0}{\mathrm{d}s}=0$, Then $\partial_s F = 0$ for all times, and (4.3) is satisfied. This proves the following proposition:

Proposition 4.1 (Parameterisation determines λ, [32])**.** *Suppose that $\lambda(0,t)=0$, and that $z=z(s,t)$ is a smooth solution to the sharp front equation (4.1) with initial condition $z_0 = z_0(s)$ parameterised so that $\partial_s z_0\cdot\partial_s^2 z_0 = 0$. Then $|z_s|^2$ is independent of s for all t (or equivalently $z_s\cdot z_{ss}=0$) if and only if λ is given by (4.7).*

4.2 Computing derivatives of terms

When computing derivatives in s, it is useful to use the periodicity in s to rewrite the periodic integrands in terms of s_*+s and s instead of s_* and s. This is so that the derivatives of difference quotients like $\frac{a(s+s_*)-a(s)}{|b(s+s_*)-b(s)|^\beta}$ retain their difference quotient structure.

In the interest of brevity, let us introduce notation for a finite difference,

$$\mathcal{D}a(s, s_*) := a(s+s_*) - a(s)$$

Using this notation, the relevant quantities (suppressing the time variable) for $s \in [0, 1)$ are

$$\begin{aligned}
\zeta(s) &= -\int_{\mathbb{T}} \frac{z_s(s+s_*) - z_s(s)}{|z_s(s+s_*) - z_s(s)|^\beta} \,\mathrm{d}s_* \\
&= -\int_{\mathbb{T}} \frac{\mathcal{D}z_s(s, s_*)}{|\mathcal{D}z(s, s_*)|^\beta} \,\mathrm{d}s_*, && (4.8) \\
\lambda(s) &= s \int_{\mathbb{T}} \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) \,\mathrm{d}s_1 - \int_0^s \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) \,\mathrm{d}s_1. && (4.9)
\end{aligned}$$

Note carefully that λ is a periodic function in s despite the explicit appearance of s and $\int_0^s \ldots \mathrm{d}s_1$ in the formula for λ. Also, note that ζ is smooth if z is, by differentiating under the integral. Therefore, the first two derivatives of ζ are (writing $\mathcal{D}h$ for $\mathcal{D}h(s, s_*)$)

$$\begin{aligned}
\zeta_s(s) &= -\int_{\mathbb{T}} \frac{\mathcal{D}z_{ss}}{|\mathcal{D}z|^\beta} - \beta \frac{\mathcal{D}z_s(\mathcal{D}z \cdot \mathcal{D}z_s)}{|\mathcal{D}z|^{\beta+2}} \,\mathrm{d}s_*, && (4.10) \\
\zeta_{ss}(s) &= -\int_{\mathbb{T}} \Bigg(\frac{\mathcal{D}z_{sss}}{|\mathcal{D}z|^\beta} - 2\beta \frac{\mathcal{D}z_{ss}(\mathcal{D}z \cdot \mathcal{D}z_s)}{|\mathcal{D}z|^{\beta+2}} \\
&\qquad - \beta \frac{\mathcal{D}z_s(|\mathcal{D}z_s|^2 + \mathcal{D}z \cdot \mathcal{D}z_{ss})}{|\mathcal{D}z|^{\beta+2}} \\
&\qquad + \beta(\beta+2) \frac{\mathcal{D}z_s(\mathcal{D}z \cdot \mathcal{D}z_s)^2}{|\mathcal{D}z|^{\beta+4}} \Bigg) \,\mathrm{d}s_*, && (4.11)
\end{aligned}$$

and the derivatives of λ are

$$\begin{aligned}
\lambda_s(s) &= \int_{\mathbb{T}} \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) \,\mathrm{d}s_1 - \frac{z_s(s)}{L^2} \cdot \zeta_s(s), && (4.12) \\
\lambda_{ss}(s) &= -\frac{z_s(s) \cdot \zeta_{ss}(s)}{L^2} - \frac{z_{ss}(s) \cdot \zeta_s(s)}{L^2}. && (4.13)
\end{aligned}$$

We have computed all the terms on the right hand side of the evolution equations

$$\begin{aligned}
z_t &= \zeta + \lambda z_s, && (4.14) \\
z_{st} &= \zeta_s + \lambda_s z_s + \lambda z_{ss}, && (4.15) \\
z_{sst} &= \zeta_{ss} + \lambda_{ss} z_s + 2\lambda_s z_{ss} + \lambda z_{sss}. && (4.16)
\end{aligned}$$

As we briefly mentioned, we also need to assume that the initial curve z_0 does not self-intersect so that the integral $\zeta(s,t)$ makes sense. In analogy with [32], let us introduce the arc-chord condition via the function $\Gamma = \Gamma(z)$ below. It acts as a quantitative control on curve length (since $\Gamma(0,0) = \frac{1}{L}$) and self-intersection, and is a slight variant of the function F introduced in [32]. The first part of the piecewise definition of $\Gamma(z)$ below diverges precisely at a self-intersection (i.e. $z(s+s_*) = z(s)$ for $s_* \neq 0$), and the second part diverges if the first derivative vanishes.

Definition 4.2 (Arc-chord condition)**.** We say that a curve z satisfies the arc chord condition if the function $\Gamma(z) : \mathbb{T}^2 \to \mathbb{R}$ defined by

$$\Gamma(z)(s,s_*) := \begin{cases} \dfrac{|\sin(\pi s_*)|}{\pi|z(s+s_*) - z(s)|} & s_* \neq 0, \\ \dfrac{1}{|z_s(s)|} & s_* = 0, \end{cases} \tag{4.17}$$

belongs to $L^\infty(\mathbb{T}^2)$.

This next lemma formalises the intuition of the paragraph preceding the above definition.

Lemma 4.3 (Analyticity of Γ)**.** *Suppose z is an analytic curve with uniform speed parameterisation $s \in \mathbb{T}$, and $\Gamma(z) \in L^\infty(\mathbb{T}^2)$. Then z does not self-intersect, and $\Gamma(z)$ is analytic on $\mathbb{T}^2$. In addition, $\Gamma(z)$ is bounded away from 0.*

Proof. Notice that $Z := \frac{z(s+s_*)-z(s)}{\sin(\pi s_*)/\pi}$ is analytic and does not vanish. The Euclidean norm of Z is therefore analytic, and the fact that this avoids zero since $\Gamma(z) \in L^\infty$ means that its reciprocal is also analytic. □

To simplify the presentation slightly, we define

$$\operatorname{Sin} s := \sin(\pi s)/\pi. \tag{4.18}$$

The goal is to apply a Cauchy–Kowalevskaya type argument, as in [27]. We will need to carefully write the evolution equations for

$$z, z_s, z_{ss}, \text{ and } \Gamma,$$

in terms of analytic functions involcing at most first order operators of the quantities above. To this end, we will use the following version of the Fundamental Theorem of Calculus.

Lemma 4.4. *Define for $s, s_* \in \mathbb{T}$, and curves w, the function $\mathcal{I}(w) = \mathcal{I}(w)(s, s_*)$ by*

$$\mathcal{I}(w)(s, s_*) := \int_0^1 w(s + (1-\tau)s_*)\,\mathrm{d}\tau, \tag{4.19}$$

Then if z is a C^1 curve,

$$\mathcal{D}z(s, s_*) = \mathcal{I}(z_s)(s, s_*)s_*.$$

Proof. This follows from the usual Fundamental Theorem of Calculus,

$$f(s + s_*) - f(s) = \int_s^{s+s_*} f'(\sigma)\,\mathrm{d}\sigma,$$

by the change of variables $\sigma = s + (1-\tau)s_*$. □

It will be convenient to extend the definition of $\mathcal{I}$ to two-variable functions $F = F(s, s_*)$ by the formula

$$\mathcal{I}(F)(s, s_*) := \int_0^1 F(s, (1-\tau)s_*)\,\mathrm{d}\tau.$$

This agrees with the previous definition (4.19) in the sense that if $F(s, s_*) = z(s+s_*)$, then $\mathcal{I}(F)(s, s_*) = \mathcal{I}(z)(s, s_*)$.

Proposition 4.5 (Expansion of Γ)**.** *The function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$ as defined in (4.17) satisfies the following first order expansion in s_*,*

$$\Gamma(s, s_*, t) = \frac{1}{L(t)} + \mathcal{I}(\partial_{s_*}\Gamma)(s, s_*)s_*,$$

where explicitly, if $s_* \neq 0$ (ignoring the t variable),

$$\partial_{s_*}\Gamma(s, s_*) = \begin{cases} -\Gamma(s, s_*)^3 \frac{\mathcal{D}z}{\operatorname{Sin} s_*} \cdot \left(\frac{z_s(s+s_*)\operatorname{Sin} s_* - \mathcal{D}z(s,s_*)\cos(\pi s_*)}{(\operatorname{Sin} s_*)^2} \right) & s_* \neq 0, \\ 0 & s_* = 0. \end{cases}$$

Proof. The formula for $\mathcal{I}(\partial_{s_*}\Gamma)$ comes from applying the Fundamental Theorem of Calculus to expand $\tilde{\Gamma}(s_*) := \Gamma(s, s_*)$ around $s_* = 0$ for fixed s,

$$\tilde{\Gamma}(s_*) = \tilde{\Gamma}(0) + \int_0^{s_*} \tilde{\Gamma}'(s_* - \tilde{\tau})\,\mathrm{d}\tilde{\tau} = \tilde{\Gamma}(0) + \int_0^1 \tilde{\Gamma}'((1-\tau)s_*)\,\mathrm{d}\tau s_*.$$

Since $\tilde{\Gamma}(0) = \frac{1}{L}$, we just need to compute $\tilde{\Gamma}'$. We have

$$\begin{aligned}
\tilde{\Gamma}'(s_*) = \partial_{s_*}\Gamma(s, s_*) &= \partial_{s_*}\left(\left|\frac{\mathcal{D}z(s,s_*)}{\operatorname{Sin} s_*}\right|^{-1}\right) \\
&= -\left|\frac{\mathcal{D}z(s,s_*)}{\operatorname{Sin} s_*}\right|^{-3}\frac{\mathcal{D}z(s,s_*)}{\operatorname{Sin} s_*}\cdot\left(\partial_{s_*}\frac{\mathcal{D}z(s,s_*)}{\operatorname{Sin} s_*}\right) \\
&= -\Gamma(s,s_*)^3\frac{\mathcal{D}z(s,s_*)}{\operatorname{Sin} s_*}\cdot\frac{(\operatorname{Sin} s_*)z_s(s+s_*) - \cos(\pi s_*)\mathcal{D}z(s,s_*)}{(\operatorname{Sin} s_*)^2},
\end{aligned}$$

as claimed[1]. For the behaviour as $s_* \to 0$, writing $\operatorname{Sin} s_* = s_* + O(s_*^3)$, and $\cos(\pi s_*) = 1 - \pi^2 s_*^2 + O(s_*^4)$,

$$\begin{aligned}
&\partial_{s*}\Gamma(s,s_*) \\
&= -\Gamma(s,0)^3 z_s(s)\cdot\frac{z_s(s+s_*)s_* - (1-\pi^2 s_*^2/2)\mathcal{D}z(s,s_*)}{s_*^2} + o(1) && (4.20) \\
&= -\Gamma(s,0)^3 z_s(s)\cdot\frac{z_s(s+s_*)s_* - (z_s(s)s_* + z_{ss}(s)s_*^2/2 + o(s_*^2))}{s_*^2} + o(1) && (4.21) \\
&= -\Gamma(s,0)^3 z_s(s)\cdot\frac{\mathcal{D}z_s(s,s_*)s_* - z_{ss}(s)s_*^2/2}{s_*^2} + o(1) \\
&= -\Gamma(s,0)^3 z_s(s)\cdot\left(z_{ss}(s) - \frac{z_{ss}(s)}{2} + o(1)\right) + o(1) && (4.22) \\
&= o(1),
\end{aligned}$$

where in going from (4.20) to (4.21) we used a Taylor expansion of z to rewrite $\mathcal{D}z$, and at line (4.22) we used $z_s \cdot z_{ss} = 0$ to see that it vanishes at $s_* = 0$. □

An exactly analogous calculation gives the following.

Lemma 4.6 (Expansion of Γ^β)**.**

$$\Gamma^\beta(s,s_*) = \frac{1}{L(t)^\beta} + \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s*}\Gamma)(s,s_*).$$

The Fundamental Theorem of Calculus will also allow us to rewrite the following dot products that appear in (4.14), (4.15), and (4.16), because of (4.10),(4.11):

$$\begin{aligned}
\mathcal{D}z\cdot\mathcal{D}z_s &= \mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss})s_*^2, && (4.23) \\
|\mathcal{D}z_s|^2 &= |\mathcal{I}(z_{ss})|^2 s_*^2, && (4.24) \\
\mathcal{D}z\cdot\mathcal{D}z_{ss} &= \mathcal{I}(z_s)\cdot\mathcal{I}(z_{sss})s_*^2. && (4.25)
\end{aligned}$$

[1]Note that factors of π have been absorbed by the definition of Sin in (4.18).

4.2.1 Expanding the integral terms $\partial_s^n \zeta$

Here we record the use of the Fundamental Theorem of Calculus (Lemma 4.19). Starting from (4.8) (4.10),and (4.11) we replace the finite differences $\mathcal{D}\partial_s^k z$ with the integral terms $\mathcal{I}(\partial_s^{k+1} z) s_*$ via (4.23),(4.24), and (4.25). Then we rewrite $|\mathcal{D}z(s,s_*)|^{-1} = |\operatorname{Sin} s_*|^{-1}\Gamma(s,s_*)$. The result of these substitutions is as follows, which shows that the terms ζ, ζ_s are given as well-behaved integrals depending on Γ and at most three derivatives of ζ.

$$\zeta(s) = -\int_{\mathbb{T}} \Gamma(s,s_*)^\beta \frac{\mathcal{I}(z_{ss}) s_*}{|\operatorname{Sin} s_*|^\beta} \, \mathrm{d}s_*, \tag{4.26}$$

$$\begin{aligned} \zeta_s(s) = &-\int_{\mathbb{T}} \Gamma(s,s_*)^\beta \frac{\mathcal{I}(z_{sss}) s_*}{|\operatorname{Sin} s_*|^\beta} \, \mathrm{d}s_*, \\ &+ \beta \int_{\mathbb{T}} \Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss})(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss})) s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \, \mathrm{d}s_*. \end{aligned} \tag{4.27}$$

For the highest derivative ζ_{ss}, to prepare to apply the abstract Cauchy–Kowalevskaya theorem, we additionally use the expansion of Γ^β in Lemma 4.6 to find a convolution operator of order higher than one.

$$\begin{aligned} \zeta_{ss}(s) = &-\frac{1}{L^\beta} \int_{\mathbb{T}} \frac{z_{sss}(s+s_*) - z_{sss}(s)}{|\operatorname{Sin} s_*|^\beta} \, \mathrm{d}s_* \\ &- \int_{\mathbb{T}} \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s,s_*) s_* \frac{z_{sss}(s+s_*) - z_{sss}(s)}{|\operatorname{Sin} s_*|^\beta} \, \mathrm{d}s_* \\ &+ 2\beta \int_{\mathbb{T}} \Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(z_{sss})(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss})) s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \, \mathrm{d}s_* \\ &+ \beta \int_{\mathbb{T}} \Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss})(|\mathcal{I}(z_{ss})|^2 + \mathcal{I}(z_s)\cdot\mathcal{I}(z_{sss})) s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \, \mathrm{d}s_* \\ &- \beta(\beta+2) \int_{\mathbb{T}} \Gamma(s,s_*)^{\beta+4} \frac{\mathcal{I}(z_{ss})(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss}))^2 s_*^5}{|\operatorname{Sin} s_*|^{\beta+4}} \, \mathrm{d}s_*. \end{aligned} \tag{4.28}$$

4.2.2 Rewriting λ and its derivatives

The term $z_s \cdot \zeta_s$ appears in the definition (4.9) for λ. We can expand this using (4.27) as

$$\begin{aligned} z_s \cdot \zeta_s = \int_{\mathbb{T}} &-\Gamma(s,s_*)^\beta \frac{\mathcal{I}(z_{sss}) \cdot z_s s_*}{|\operatorname{Sin} s_*|^\beta} \\ &+ \beta \Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_s (\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss})) s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \, \mathrm{d}s_*. \end{aligned}$$

Since $z_s \cdot z_{ss} = 0$, Lemma 4.19 gives that

$$\mathcal{I}(z_{sss}) \cdot z_s(s)s_* + \mathcal{I}(z_{ss}) \cdot z_{ss}(s+s_*)s_* = 0.$$

Indeed,

$$\begin{aligned}
\mathcal{I}(z_{sss}) \cdot z_s(s)s_* &= z_{ss}(s+s_*) \cdot z_s(s) \\
&= z_{ss}(s+s_*) \cdot (z_s(s+s_*) - \mathcal{I}(z_{ss})s_*) \\
&= -z_{ss}(s+s_*) \cdot \mathcal{I}(z_{ss})s_*.
\end{aligned}$$

So we can remove completely the dependence on the third derivative:

$$\begin{aligned}
z_s \cdot \zeta_s = \int_{\mathbb{T}} &\Gamma(s,s_*)^{\beta} \frac{\mathcal{I}(z_{ss}) \cdot z_{ss}(s+s_*)s_*}{|\operatorname{Sin} s_*|^{\beta}} \\
&+ \beta\Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_s(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \, \mathrm{d}s_*.
\end{aligned}$$

Hence, we can write the equation (4.9) for λ as (using $\frac{1}{L} = \Gamma(s,0) = \Gamma(0,0)$)

$$\begin{aligned}
&\lambda(s) \\
&= s\int_{\mathbb{T}} \frac{z_s(s_1)}{|z_s(s_1)|^2} \cdot \zeta_s(s_1)\,\mathrm{d}s_1 - \int_0^s \frac{z_s(s_1)}{|z_s(s_1)|^2} \cdot \zeta_s(s_1)\,\mathrm{d}s_1 \\
&= \frac{s}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Bigg(\Gamma(s_1,s_0)^{\beta} \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_{ss}(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} + \beta\Gamma(s_1,s_0)^{\beta+2} \times \\
&\quad \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1,s_0) \cdot \mathcal{I}(z_{ss})(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\Bigg)\,\mathrm{d}s_0\,\mathrm{d}s_1 \\
&\quad - \frac{1}{L^2}\int_0^s\int_{\mathbb{T}} \Bigg(\Gamma(s_1,s_0)^{\beta} \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_{ss}(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} + \beta\Gamma(s_1,s_0)^{\beta+2} \times \\
&\quad \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1,s_0) \cdot \mathcal{I}(z_{ss})(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\Bigg)\,\mathrm{d}s_0\,\mathrm{d}s_1.
\end{aligned} \tag{4.29}$$

(We have used $\times$ to denote scalar multiplication.) Similarly, we can rewrite (4.12), which is the equation for λ_s as

$$\begin{aligned}
&\lambda_s(s) \\
&= \frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Bigg(\Gamma(s_1,s_0)^{\beta} \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_{ss}(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} + \beta\Gamma(s_1,s_0)^{\beta+2} \times \\
&\quad \frac{\mathcal{I}(z_{ss})(s_1,s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1,s_0) \cdot \mathcal{I}(z_{ss})(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\Bigg)\,\mathrm{d}s_0\,\mathrm{d}s_1
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{L^2}\int_{\mathbb{T}}\Bigg(\Gamma(s,s_*)^\beta\frac{\mathcal{I}(z_{ss})(s,s_*)\cdot z_{ss}(s+s_*)s_*}{|\operatorname{Sin} s_*|^\beta}+\beta\Gamma(s,s_*)^{\beta+2}\times\\
&\frac{\mathcal{I}(z_{ss})(s,s_*)\cdot z_s(s)(\mathcal{I}(z_s)(s,s_*)\cdot\mathcal{I}(z_{ss})(s,s_*))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\Bigg)\,\mathrm{d}s_*. \qquad (4.30)
\end{aligned}
$$

The final term to rewrite involving λ is λ_{ss}, which is (copying (4.13))

$$
\lambda_{ss}(s)=-\frac{1}{L^2}\partial_s(\zeta_s\cdot z_s)=-\frac{1}{L^2}(\zeta_{ss}\cdot z_s+\zeta_s\cdot z_{ss}).
$$

That this is better behaved than ζ_{ss} can be seen from the identities x obtained from the cancellation $z_s\cdot z_{ss}=0$,

$$
\begin{aligned}
z_{sss}(s)\cdot z_s(s)&=-|z_{ss}(s)|^2,\\
z_{sss}(s+s_*)\cdot z_s(s)&=-|z_{ss}(s+s_*)|^2-z_{sss}(s+s_*)\cdot\mathcal{I}(z_{ss})(s,s_*)s_*,
\end{aligned}
$$

which means that

$$
\begin{aligned}
&\mathcal{D}z_{sss}(s,s_*)\cdot z_s(s)\\
&=-\mathcal{D}|z_{ss}|^2(s,s_*)-z_{sss}(s+s_*)\cdot\mathcal{I}(z_{ss})(s,s_*)s_*\\
&=\big(-2\mathcal{I}(z_{sss}\cdot z_{ss})(s,s_*)-z_{sss}(s+s_*)\cdot\mathcal{I}(z_{ss})(s,s_*)\big)s_*.
\end{aligned}
$$

Therefore, λ_{ss} depends on Γ and the first three derivatives of z. For reference, the full expansion of λ_{ss} is as follows, which comes from a similar derivation to that of (4.28) but the cancellations above are used to regularise the integral, in place of Lemma 4.6.

$$
\begin{aligned}
\lambda_{ss}(s)=\frac{1}{L^2}\int_{\mathbb{T}}&\Gamma(s,s_*)^\beta\frac{(-2\mathcal{I}(z_{sss}\cdot z_{ss})-z_{sss}(s+s_*)\cdot\mathcal{I}(z_{ss}))s_*}{|\operatorname{Sin} s_*|^\beta}\\
&-2\beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(z_{sss})\cdot z_s(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss}))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\\
&-\beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(z_{ss})\cdot z_s(|\mathcal{I}(z_{ss})|^2+\mathcal{I}(z_s)\cdot\mathcal{I}(z_{sss}))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\\
&+\beta(\beta+2)\Gamma(s,s_*)^{\beta+4}\frac{\mathcal{I}(z_{ss})\ \cdot z_s(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss}))^2s_*^5}{|\operatorname{Sin} s_*|^{\beta+4}}\\
&+\Gamma(s,s_*)^\beta\frac{\mathcal{I}(z_{sss})\cdot z_{ss}s_*}{|\operatorname{Sin} s_*|^\beta}\\
&-\beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(z_{ss})\cdot z_{ss}(\mathcal{I}(z_s)\cdot\mathcal{I}(z_{ss}))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_*. \qquad (4.31)
\end{aligned}
$$

4.3 Rewriting the evolution equations

Define the functions f, g and h by

$$f(s,t) = z(s,t), \quad g(s,t) = z_s(s,t), \quad h(s,t) = z_{ss}(s,t).$$

4.3.1 Evolution of f

Rewriting $z_t = \zeta + \lambda z_s$ (4.14) using (4.26) and (4.29), we obtain

$$\begin{aligned}
&f_t(s)\\
&= -\int_{\mathbb{T}} \Gamma(s,s_*)^\beta \frac{\mathcal{I}(h)(s,s_*)s_*}{|\operatorname{Sin} s_*|^\beta}\,\mathrm{d}s_*\\
&\quad + g(s)\Bigg(s\frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Gamma(s_1,s_0)^\beta \frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^\beta} + \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\qquad \frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\\
&\qquad - \frac{1}{L^2}\int_0^s\int_{\mathbb{T}} \Gamma(s_1,s_0)^\beta \frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^\beta} + \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\qquad \frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\Bigg).
\end{aligned} \tag{4.32}$$

4.3.2 Evolution of g

Rewriting $z_{st} = \zeta_s + \lambda z_{ss} + \lambda_s z_s$ (4.15) using (4.27),(4.29), and (4.30), we obtain

$$\begin{aligned}
g_t(s) &= -\int_{\mathbb{T}} \Gamma(s,s_*)^\beta \frac{\mathcal{I}(h_s)s_*}{|\operatorname{Sin} s_*|^\beta}\,\mathrm{d}s_*\\
&\quad + \beta\int_{\mathbb{T}} \Gamma(s,s_*)^{\beta+2} \frac{\mathcal{I}(h)(\mathcal{I}(g)\cdot\mathcal{I}(h))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_*\\
&\quad + h(s)\Bigg(s\frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Gamma(s_1,s_0)^\beta \frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^\beta} + \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\qquad \frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\\
&\qquad - \frac{1}{L^2}\int_0^s\int_{\mathbb{T}} \Gamma(s_1,s_0)^\beta \frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^\beta} + \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\qquad \frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\Bigg)\\
&\quad + g(s)\Bigg(\frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Gamma(s_1,s_0)^\beta \frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^\beta} + \beta\Gamma(s_1,s_0)^{\beta+2}\times
\end{aligned}$$

$$
\frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1
- \frac{1}{L^2}\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta}\frac{\mathcal{I}(h)(s,s_*)\cdot h(s+s_*)s_*}{|\operatorname{Sin} s_*|^{\beta}} + \beta\Gamma(s,s_*)^{\beta+2}\times
\frac{\mathcal{I}(h)(s,s_*)\cdot g(s)(\mathcal{I}(g)(s,s_*)\cdot\mathcal{I}(h)(s,s_*))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_*\Bigg). \tag{4.33}
$$

4.3.3 Evolution of h

Rewriting $z_{sst} = \zeta_{ss} + \lambda_{ss}z_s + 2\lambda_s z_{ss} + \lambda z_{sss}$ (4.16) using (4.28), (4.29), (4.30), and (4.31), we obtain

$$
\begin{aligned}
h_t = {} & \frac{-1}{L^{\beta}}\int_{\mathbb{T}}\frac{h_s(s+s_*)-h_s(s)}{|\operatorname{Sin} s_*|^{\beta}}\,\mathrm{d}s_* \\
& - \int_{\mathbb{T}}\mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s,s_*)s_*\frac{h_s(s+s_*)-h_s(s)}{|\operatorname{Sin} s_*|^{\beta}}\,\mathrm{d}s_* \\
& + 2\beta\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h_s)(\mathcal{I}(g)\cdot\mathcal{I}(h))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_* \\
& + \beta\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h)(|\mathcal{I}(h)|^2+\mathcal{I}(g)\cdot\mathcal{I}(h_s))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_* \\
& - \beta(\beta+2)\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta+4}\frac{\mathcal{I}_{2,0}(\mathcal{I}(g)\cdot\mathcal{I}(h))^2 s_*^5}{|\operatorname{Sin} s_*|^{\beta+4}}\,\mathrm{d}s_* \\
& + g(s)\Bigg(\frac{1}{L^2}\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta}\frac{(-2\mathcal{I}(h_s\cdot h)-h_s(s+s_*)\cdot\mathcal{I}(h))s_*}{|\operatorname{Sin} s_*|^{\beta}} \\
& \quad - 2\beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h)\cdot g(\mathcal{I}(g)\cdot\mathcal{I}(h))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \\
& \quad - \beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h)\cdot g(|\mathcal{I}(h)|^2+\mathcal{I}(g)\cdot\mathcal{I}(h_s))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \\
& \quad + \beta(\beta+2)\Gamma(s,s_*)^{\beta+4}\frac{\mathcal{I}_{2,0}\cdot g(\mathcal{I}(g)\cdot\mathcal{I}(h))^2 s_*^5}{|\operatorname{Sin} s_*|^{\beta+4}} \\
& \quad + \Gamma(s,s_*)^{\beta}\frac{\mathcal{I}(h_s)\cdot h s_*}{|\operatorname{Sin} s_*|^{\beta}} - \beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h)\cdot h(\mathcal{I}(g)\cdot\mathcal{I}(h))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_*\Bigg) \\
& + 2h(s)\Bigg(\frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\Gamma(s_1,s_0)^{\beta}\frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} + \beta\Gamma(s_1,s_0)^{\beta+2}\times \\
& \quad \frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1 \\
& \quad - \frac{1}{L^2}\int_{\mathbb{T}}\Gamma(s,s_*)^{\beta}\frac{\mathcal{I}(h)(s,s_*)\cdot h(s+s_*)s_*}{|\operatorname{Sin} s_*|^{\beta}}
\end{aligned}
$$

$$
\left. - \beta\Gamma(s,s_*)^{\beta+2}\frac{\mathcal{I}(h)(s,s_*)\cdot g(s)(\mathcal{I}(g)(s,s_*)\cdot\mathcal{I}(h)(s,s_*))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}}\,\mathrm{d}s_*\right)
$$

$$
\begin{aligned}
&+ h_s(s)\Bigg(s\frac{1}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\Gamma(s_1,s_0)^{\beta}\frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} + \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\\
&\quad - \frac{1}{L^2}\int_0^s\int_{\mathbb{T}}\Gamma(s_1,s_0)^{\beta}\frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin} s_0|^{\beta}} - \beta\Gamma(s_1,s_0)^{\beta+2}\times\\
&\qquad\frac{\mathcal{I}(h)(s_1,s_0)\cdot g(s_1)(\mathcal{I}(g)(s_1,s_0)\cdot\mathcal{I}(h)(s_1,s_0))s_0^3}{|\operatorname{Sin} s_0|^{\beta+2}}\,\mathrm{d}s_0\,\mathrm{d}s_1\Bigg).
\end{aligned}
\tag{4.34}
$$

4.3.4 Evolution of Γ

The function Γ defined in (4.17) evolves according to the equation

$$
\Gamma_t = \Gamma^3\frac{\mathcal{D}z}{|\operatorname{Sin} s_*|}\cdot\frac{\mathcal{D}z_t}{|\operatorname{Sin} s_*|} = \frac{\Gamma^3 s_*^2}{(\operatorname{Sin} s_*)^2}\mathcal{I}(g)\cdot\mathcal{I}(g_t). \tag{4.35}
$$

4.3.5 Summary of dependencies

We now define the operators E_1, E_2, E_3, and E_4 using the right-hand sides of the above evolution equations. That is, E_1, and E_2 are defined as the right-hand sides of (4.32) and (4.33) respectively. We do almost the same for E_3 in (4.34), except that we single out a term involving a convolution. E_4 is defined by (4.35), where we use $g_t = E_2[g,h,h_s,\Gamma]$ to rewrite the term $\mathcal{I}(g_t)$. This gives

$$
\begin{aligned}
f_t &= E_1[g,h,\Gamma],\\
g_t &= E_2[g,h,h_s,\Gamma],\\
h_t &= \frac{1}{L^{\beta}}\int_{\mathbb{T}}\frac{h_s(s+s_*)-h_s(s_*)}{|\operatorname{Sin} s_*|^{\beta}}\,\mathrm{d}s_* + E_3[g,h,h_s,\Gamma], && (4.36)\\
\Gamma_t &= E_4[\Gamma,g,F_2[g,h,h_s\Gamma]]. && (4.37)
\end{aligned}
$$

Note that in equation (4.36) for h, we have singled out a term involving

$$
\mathcal{H}_{\beta}(h) := \int_{\mathbb{T}}\frac{h_s(s+s_*)-h_s(s_*)}{|\operatorname{Sin} s_*|^{\beta}}\,\mathrm{d}s_*.
$$

Recall that the symbol of a Fourier multiplier $f\mapsto M(\partial_s)f$ is the function $M(k)$ defined in frequency space.

Lemma 4.7. *(Skew-symmetry) The symbol of $\mathcal{H}_{\beta}$ is purely imaginary.*

Proof. $\mathcal{H}_\beta$ is the convolution of h_s with the renormalised distribution of $|\sin s|^{-\beta}$ (up to possibly a multiple of the Dirac delta),

$$\langle \mathcal{R}_{|\sin s|^{-\beta}}, \phi\rangle := \int_{\mathbb{T}} \frac{\phi(s) - \phi(0)}{|\sin s|^{\beta}}\,\mathrm{d}s.$$

As $|\sin s|^{-\beta}$ is even and real valued, so is the Fourier transform $F(k) = \mathcal{F}(\mathcal{R}_{|\sin s|^{-\beta}})(k)$. Hence, the symbol of $\mathcal{H}_\beta$ is $-\mathrm{i}2\pi k F(k)$, which is purely imaginary. $\square$

Any operator $\mathrm{i}m(\partial_s)$ with purely imaginary symbol $\mathrm{i}m(k)$ has the skew-symmetry property $(\mathrm{i}m(\partial_s)h, h)_{L^2} = 0$. This can be seen from $(\mathrm{i}m(\partial_s)g, h)_{L^2(\mathbb{T})} = (\mathrm{i}m(k)\hat{g}(k), \hat{h}(k))_{\ell^2(\mathbb{Z})} = -(\hat{g}(k), \mathrm{i}m(k)\hat{h}(k))_{\ell^2(\mathbb{Z})}$. The important point is that the operator $\partial_t - \mathrm{i}m(\partial_s)$ is boundedly invertible on L^2-based Sobolev spaces (contrast this with say, the backward heat equation), with the explicit solution defined via its Fourier coefficients ($k \in \mathbb{Z}$),

$$\partial_t f - \mathrm{i}m(\partial_s)f = g \iff \hat{f}(k) = \mathrm{e}^{-\mathrm{i}m(k)t}\hat{f}_0(k) + \int_0^t \hat{g}(k)\mathrm{e}^{\mathrm{i}m(k)(\tilde{t}-t)}\,\mathrm{d}\tilde{t}.$$

For the time dependent operator $\partial_t - L(t)^{-\beta}m(\partial_s)$, we can first perform the time rescaling $\partial_{t_0} f(s, t(t_0)) = L(t)^{\beta}\partial_t f$ with $t(t_0) = \int_0^{t_0} L(\tau)^{\beta}\,\mathrm{d}\tau$. In these coordinates, the equation is $\partial_{t_0} f - \mathrm{i}m(\partial_s)f = g$. Writing $t_0(t)$ for the inverse of $t(t_0)$, applying the above formula gives

$$\begin{aligned} &\partial_t f - \mathrm{i}L(t)^{-\beta}m(\partial_s)f = g \\ \iff\ &\hat{f}(k,t) = \mathrm{e}^{-\mathrm{i}m(k)t_0(t)}\hat{f}_0(k) + \int_0^{t_0(t)} \hat{g}(k, t(\tilde{t}_0))\mathrm{e}^{\mathrm{i}m(k)(\tilde{t}_0 - t_0(t))}\,\mathrm{d}\tilde{t}_0. \end{aligned}$$

A change of variables $\tilde{t}_0 = t_0(\tilde{t})$ gives

$$\hat{f}(k,t) = \mathrm{e}^{-\mathrm{i}m(k)t_0(t)}\hat{f}_0(k) + \int_0^t L(\tilde{t})^{-\beta}\hat{g}(k,\tilde{t})\mathrm{e}^{\mathrm{i}m(k)(t_0(\tilde{t}) - t_0(t))}\,\mathrm{d}\tilde{t}. \tag{4.38}$$

We will take advantage of this to allow the use of the abstract Cauchy–Kowalevskaya theorem, despite the fact that the original CDE involves an operator of order higher than one.

4.4 Adapting the equation for Theorem 3.6

Here, we define the Banach scale that we will use (and its component Banach spaces). We identify analytic functions with their analytic continuations.

Definition 4.8. Given $l \in \mathbb{N}$ and $\rho > 0$, we say that a function $f : \mathbb{R} \to \mathbb{R}^2$ is in $K^{l,\rho}$ if

1. $f = f(s + \mathrm{i}\tilde{s})$ is 1-periodic (periodic with period 1) in $\operatorname{Re} s$ and analytic in $|\operatorname{Im} s| < \rho$.

2. For every $|\operatorname{Im} s| < \rho, \partial_s^\alpha f(\operatorname{Re} s + \mathrm{i} \operatorname{Im} s) \in L^2_{\operatorname{Re} s}(\mathbb{T})$, that is, square integrable as a periodic function of the real part only.

3. The norm $\|f\|_{K^{l,\rho}}$ is finite, where

$$\|f\|_{K^{l,\rho}} := \sum_{\alpha \le l} \sup_{|\tilde{s}|<\rho} \|\partial_x^\alpha f(s + \mathrm{i}\tilde{s})\|_{L^2_s(\mathbb{T})} .$$

Definition 4.9. Given $l \in \mathbb{N}$ and $\rho > 0$, we say that a two-variable function $f : \mathbb{R}^2 \to \mathbb{R}$ is in $K_2^{l,\rho}$ if

1. $f(s + \mathrm{i}\tilde{s}, s_* + \mathrm{i}\tilde{s}_*)$ is 1-periodic in $\tilde{s}$ and $\tilde{s}_*$, and analytic in

$$\{(s + \mathrm{i}\tilde{s}, s_* + \mathrm{i}\tilde{s}_*) \in \mathbb{C}^2 : |\tilde{s}| < \rho, |\tilde{s}_*| < \rho\}.$$

2. For every $\alpha_1 + \alpha_2 \le l$, $\max(|\tilde{s}|, |\tilde{s}_*|) < \rho, \partial_s^{\alpha_1} \partial_{s*}^{\alpha_2} f \in L^2_{s,s_*}(\mathbb{T}^2)$.

3. The norm $\|f\|_{K_2^{l,\rho}}$ is finite, where

$$\|f\|_{K_2^{l,\rho}} := \sum_{\alpha_1+\alpha_2 \le l} \sup_{\substack{|\tilde{s}|<\rho \\ |\tilde{s}_*|<\rho}} \left\| \|\partial_s^{\alpha_1} \partial_{s*}^{\alpha_2} f(s + \mathrm{i}\tilde{s}, s_* + \mathrm{i}\tilde{s}_*)\|_{L^2_s(\mathbb{T})} \right\|_{L^2_{s_*}(\mathbb{T})} .$$

The norm $\|f\|_{K_2^{l,\rho}}$ can also be written as

$$\|f\|_{K_2^{l,\rho}} = \sum_{\alpha_1+\alpha_2 \le l} \sup_{\substack{|\tilde{s}|<\rho \\ |\tilde{s}_*|<\rho}} \|\partial_s^{\alpha_1} \partial_{s*}^{\alpha_2} f(s + \mathrm{i}\tilde{s}, s_* + \mathrm{i}\tilde{s}_*)\|_{L^2_{s,s_*}(\mathbb{T}^2)} .$$

If $\mathcal{F}f(k) := \int_{\mathbb{T}} f(x) \mathrm{e}^{-2\pi \mathrm{i} k s} \,\mathrm{d}s,\ k \in \mathbb{Z}$ denotes the Fourier transform of f, then the $K^{l,\rho}$ norm is equivalent to the weighted Sobolev norm

$$\|f\|_{K^{l,\rho}} = \|\mathrm{e}^{2\pi\rho|k|}(1 + |k|^l)\mathcal{F}f(k)\|_{\ell^2_k(\mathbb{Z})},$$

which can be seen by analytic continuation in s_0 of the well-known identity for the Fourier transform

$$\mathcal{F}_x[f(s - s_0)](k) = \mathrm{e}^{-2\pi \mathrm{i} s_0 k} \mathcal{F}f(k).$$

The following standard result follows from the Sobolev Embedding Theorem[2].

Proposition 4.10 (Banach Algebra). *1. For $l \geq 1$, $K^{l,\rho}$ is a Banach algebra:*

$$\|uv\|_{K^{l,\rho}} \lesssim_{l,\rho} \|u\|_{K^{l,\rho}} \|v\|_{K^{l,\rho}}.$$

2. For $l \geq 2$, $K_2^{l,\rho}$ is a Banach algebra:

$$\|uv\|_{K_2^{l,\rho}} \lesssim_{l,\rho} \|u\|_{K_2^{l,\rho}} \|v\|_{K_2^{l,\rho}}.$$

We now choose the Banach scale X_ρ to apply Theorem 3.6. The purpose of l is only to obtain the above Banach Algebra property, and is fixed to be any number $l \geq 2$. We now use the spaces $K^{l,\rho}$ and $K_2^{l,\rho}$ to define our Banach scale. The choice is made so that a representative element of X_ρ will be something like (z, z_s, z_{ss}, Γ) (see (4.47)).

Definition 4.11. Let $l \geq 2$ be arbitrary but fixed. The Banach scale X_ρ is

$$X_\rho := K^{l,\rho} \times K^{l,\rho} \times K^{l,\rho} \times K_2^{l,\rho}.$$

4.4.1 Application of the abstract Cauchy–Kowalevskaya Theorem

In order to apply Theorem 3.6, we will need to:

Step 1. Rewrite the evolution equations so that the evolution begins from zero initial data, which allows the iteration to begin using the estimate (CK2).

Step 2. Rewrite the equations in a suitable integral form that satisfies the continuity assumption (CK1) and the required Cauchy estimate (CK3).

We now implement these steps.

Step 1

Define the initial conditions

$$\begin{aligned} f_0(s) &= f(s,0), \\ g_0(s) &= g(s,0), \\ h_0(s) &= h(s,0), \end{aligned}$$

[2] Briefly, we need enough weak derivatives in L^2 so that their L^∞ norms are controlled, see for instance [24].

$$\Gamma_0(s,s_*) = \Gamma(s,s_{,*},0).$$

Then define the new variables $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\Gamma}$ by

$$\begin{aligned}
\tilde{f}(s,t) &= f(s,t) - f(s,0), \\
\tilde{g}(s,t) &= g(s,t) - g(s,0), \\
\tilde{h}(s,t) &= h(s,t) - h(s,0), \\
\tilde{\Gamma}(s,s_*,t) &= \Gamma(s,s_*,t) - \Gamma(s,s_*0).
\end{aligned}$$

Then the evolution equations in terms of $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\Gamma}$ are

$$\tilde{f}_t = E_1[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{\Gamma} + \Gamma_0], \tag{4.39}$$

$$\tilde{g}_t = E_2[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h_0', \tilde{\Gamma} + \Gamma_0], \tag{4.40}$$

$$\begin{aligned}
\tilde{h}_t &= (\tilde{\Gamma}(0,0) + \Gamma_0(0,0))^{\beta} \mathcal{H}_{\beta}(\tilde{h} + h_0) \\
&\quad + E_3[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h_0', \tilde{\Gamma} + \Gamma_0],
\end{aligned} \tag{4.41}$$

$$\tilde{\Gamma}_t = E_4[\tilde{\Gamma} + \Gamma_0, \tilde{g} + g_0, F_2[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h_0', \tilde{\Gamma} + \Gamma_0]]. \tag{4.42}$$

Step 2

The strategy is to integrate the equations for $\tilde{f}, \tilde{g}, \tilde{\Gamma}$ in time, and invert the operator $(\partial_t - (\tilde{\Gamma}(0,0) + \Gamma_0(0,0))\mathcal{H}_{\beta}$ for the h equation. If we define the vector of functions u and initial conditions u_0 by

$$u(s,s_*,t) := \begin{pmatrix} \tilde{f}(s,t) \\ \tilde{g}(s,t) \\ \tilde{h}(s,t) \\ \tilde{\Gamma}(s,s_*,t) \end{pmatrix}, \; u_0(s,s_*) = \begin{pmatrix} f_0(s) \\ g_0(s) \\ h_0(s) \\ \Gamma_0(s,s_*) \end{pmatrix},$$

then we can write the evolution equations as

$$\begin{aligned}
u &= F[u], \\
F[u] &= \begin{pmatrix} F_1[u_0,u] \\ F_2[u_0,u] \\ F_3[u_0,u,\nabla u] \\ F_4[u_0,u] \end{pmatrix},
\end{aligned} \tag{4.43}$$

where $\nabla u = \nabla_{s,s_*} u$ is the spatial gradient in s and s_*, and the component operators F_i of $F : X_\rho \to \cup_{\rho>0} X_\rho$ are

$$\begin{aligned}
F_1 &:= f_0 + \int_0^t E_1 \, \mathrm{d}t, \\
F_2 &:= g_0 + \int_0^t E_2 \, \mathrm{d}t, \\
F_3 &:= \Big((\partial_t - (\Gamma_0(0,0,t) + \tilde{\Gamma}(0,0))\mathcal{H}_\beta \Big)^{-1} E_3, \\
F_4 &:= \Gamma_0 + \int_0^t E_4 \, \mathrm{d}t.
\end{aligned} \tag{4.44}$$

(The omitted inputs of F_i are as in (4.43), and the omitted inputs of E_i are as in (4.39), (4.40), (4.41), and (4.42).) The inverse operator in (4.44) is defined by (4.38). This completes the derivation of the equation to which Theorem 3.6 can be applied: it only remains to check that F satisfies the assumptions of Theorem 3.6.

4.4.2 Estimates

In this section, we give some estimates and explain how they are used to show that our system satisfies (CK3).

We will show the result for a few representative terms that illustrate the method. All remaining terms follow analogously.

We write $U_1 = U_1(u), U_2 = U_2(u), \ldots$ to denote the images of u under any of the following well behaved operators: $U_i(u) = u$, $U_i(u)(s, s_*) = u(s + s_*, s + s_*)$, $U_i(u) = u(c, c_*)$, or $U_i(u) = \mathcal{I}(u)$. For any collection of M such operators $U_1, \ldots, U_M$ ($M \geq 1$), we write $\mathbf{U}_M(u) = (U_1(u), \ldots, U_M(u))$ for the function that takes values in $\mathbb{C}^{M'}$, where M' is an integer depending on M and the choices of U_i.

The operator $F_1[u_0, u]$ is a sum of time integrals of products of terms of the form

$$\Phi(\mathbf{U}_M(u)(s, s_*)),$$

where $\Phi : \mathbb{C}^{M'} \to \mathbb{C}^N$ is analytic, or terms of the form

$$\int_{\mathbb{T}} \Phi(\mathbf{U}_M(u)(s, s_*)) \, \mathrm{d}\mu(s_*),$$

where μ is a finite measure in s_*, or terms of the form

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \Phi(\mathbf{U}_M(u)(s, s_*)) \, \mathrm{d}\mu(s_*) \, \mathrm{d}s,$$

or terms of the form

$$\left(s\int_{\mathbb{T}}-\int_0^s\right)\int_{\mathbb{T}}\Phi(\mathbf{U}_M(u)(s_0,s_1))\,\mathrm{d}\mu(s_1)\,\mathrm{d}s_0.$$

This last type of term clearly creates a new periodic analytic function from an old one and presents no new issues (see Example 4.14.) For each one of the other terms, we have the following elementary lemmas.

Lemma 4.12 (Triangle inequality for time integral)**.** *For any function $u\in X_\rho$, $\|\int_0^t u\,\mathrm{d}t\|_\rho\le\int_0^t\|u\|_\rho\,\mathrm{d}t$.*

Lemma 4.13 (Local Lipschitz estimates)**.** *Let $\Phi(x_1,\dots,x_{M'})$ be analytic on an open set containing the set $A=\{x:\sum_{i=1}^{M'}|x_i|\le R\}$. Then Φ is locally Lipschitz on A,*

$$|\Phi(x_1,\dots,x_{M'})-\Phi(y_1,\dots,y_{M'})|\lesssim_{R,\Phi}\sum_{i=1}^{M'}|x_i-y_i|.$$

For $\|u\|_\rho\le R$, we have the estimate

$$\|\Phi(\mathbf{U}_M(u_1))-\Phi(\mathbf{U}_M(u_2))\|_{K_2^{l,\rho}}\lesssim_{R,\Phi,l}\|u_1-u_2\|_\rho.$$

If in addition μ is a finite measure on $\mathbb{T}$, then we have the estimates

$$\left\|\int_{\mathbb{T}}\Phi(\mathbf{U}_M(u_1))(\cdot,s_*)-\Phi(\mathbf{U}_M(u_2))(\cdot,s_*)\,\mathrm{d}\mu(s_*)\right\|_{K^{l,\rho}}\lesssim_{R,\Phi,l,\mu}\|u_1-u_2\|_\rho,$$
$$\left|\int_{\mathbb{T}}\int_{\mathbb{T}}\Phi(\mathbf{U}_M(u_1))(s,s_*)-\Phi(\mathbf{U}_M(u_2))(s,s_*)\,\mathrm{d}\mu(s_*)\,\mathrm{d}s\right|\lesssim_{R,\Phi,l,\mu}\|u_1-u_2\|_\rho.$$

In our application, we will use the finite measures $\mathrm{d}\mu(s_*)=\frac{s_*^{k+1}\,\mathrm{d}s_*}{|\operatorname{Sin}s_*|^{\beta+k}}$, for some $k>0$.

Example 4.14. In E_1, the following terms appear (see (4.32)):

$$\begin{aligned}\tilde{E}_1(u):=g(s)\frac{s}{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\Gamma(s_1,s_0)^\beta\frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin}s_0|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1\\-g(s)\frac{1}{L^2}\int_0^s\int_{\mathbb{T}}\Gamma(s_1,s_0)^\beta\frac{\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)s_0}{|\operatorname{Sin}s_0|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1.\end{aligned}$$

The first term is (at least away from $s=0,1$ before periodising) a product of the analytic function of u and $u(0,0)$, $g(s)\frac{s}{L^2}$ with the double integral against $\frac{s_0}{|\operatorname{Sin}s_0|^\beta}\,\mathrm{d}s_0\,\mathrm{d}s_1$ of the analytic function

$$\Phi(u(s_1,s_0),\mathcal{I}(u)(s_1,s_0),u(s_0+s_1,s_0+s_1))=\Gamma(s_1,s_0)^\beta\mathcal{I}(h)(s_1,s_0)\cdot h(s_0+s_1)$$

The second term is similar, and together with the first term, gives the analyticity at $s = 0, 1$ as well. Therefore, by Lemma 4.13, we have

$$\|\tilde{E}_1(u_1) - \tilde{E}_1(u_2)\|_{K^{l,\rho}} \lesssim \|u_1 - u_2\|_\rho,$$

which after integrating in time, is better than the required estimate (CK3). The other terms in E_1 are similar.

For F_2, most of the terms are also treated in a different way, except one which involves h_s. For this term, we will use the following Cauchy-type estimate.

Lemma 4.15 (Cauchy estimate)**.** *For any $l \geq 0, \rho \geq 0$, $\rho' \in (0, \rho)$, we have*

$$\|\nabla u\|_{\rho'} \leq \frac{C}{\rho - \rho'} \|u\|_\rho.$$

Proof. This proof closely follows the proof of the more elementary result without a scale of Banach spaces (c.f. (3.21)). It is enough to prove this for the spaces $K^{l,\rho}$ and $K_2^{l,\rho}$. We give the proof for $K^{l,\rho}$, since $K_2^{l,\rho}$ can be treated in exactly the same way. That is, we shall prove for $u \in K^{l,\rho}$,

$$\|\partial_s u\|_{K^{l,\rho'}} \leq \frac{C}{\rho - \rho'} \|u\|_{K^{l,\rho}}.$$

From the definition of the $K^{l,\rho}$ norm in Definition 4.8, it suffices to prove that for every $v = \partial_s^r u$, $r = 0, 1, \ldots, l$,

$$\sup_{|\tilde{s}| < \rho'} \|\partial_s v(s + \mathrm{i}\tilde{s})\|_{L_s^2} \leq \frac{C}{\rho - \rho'} \sup_{|\tilde{s}| < \rho} \|v(s + \mathrm{i}\tilde{s})\|_{L_s^2}, \tag{4.45}$$

Set $0 < \delta < \rho - \rho'$. Then the well-known Cauchy Integral Formula for a derivative gives for $z = s + \mathrm{i}s \in \mathbb{C}$, $|\tilde{s}| < \rho'$,

$$\partial_s v(s + \mathrm{i}\tilde{s}) = \frac{1}{2\pi\mathrm{i}} \int_{|z-w|=\delta} \frac{v(w)}{(z-w)^2} \,\mathrm{d}w = \frac{1}{2\pi\mathrm{i}} \int_{|w|=\delta} \frac{v(s + \mathrm{i}\tilde{s} + w)}{w^2} \,\mathrm{d}w.$$

Taking the L_s^2 norm and using the periodicity of v in the real part to obtain (4.46),

$$\begin{aligned}
\|\partial_s v(s + \mathrm{i}\tilde{s})\|_{L_s^2} &\leq \frac{1}{2\pi} \int_{|w|=\delta} \frac{\|v(s + \mathrm{i}\tilde{s} + w)\|_{L_s^2}}{|w|^2} \,\mathrm{d}l(w) \\
&= \frac{1}{2\pi} \int_{|w|=\delta} \frac{\|v(s + \mathrm{i}(\tilde{s} + \operatorname{Im} w))\|_{L_s^2}}{|w|^2} \,\mathrm{d}l(w) \qquad (4.46)
\end{aligned}$$

$$\leq \frac{1}{2\pi\delta^2} \int_{|w|=\delta} \mathrm{d}l(w) \sup_{|\tilde{\tilde{s}}|<\rho'+\delta} \|v(s+\mathrm{i}\tilde{\tilde{s}})\|_{L_s^2}$$
$$\leq \frac{1}{\delta} \sup_{|\tilde{\tilde{s}}|<\rho} \|v(s+\mathrm{i}\tilde{\tilde{s}})\|_{L_s^2},$$

where $\mathrm{d}l$ is the arc-length measure on $|w| = \delta$. Taking a limit $\delta \to \rho - \rho'$, and then a supremum over all $\tilde{s}$ with $|\tilde{s}| < \rho'$ leads to (4.45). By the earlier discussion, we have finished the proof of Lemma 4.15. □

Example 4.16. The first term of (4.33) is

$$\tilde{E}_2(u) = -\int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h_s)s_*}{|\operatorname{Sin} s_*|^\beta} \,\mathrm{d}s_*.$$

To show that this term satisfies the assumption (CK3), first use the local Lipschitz estimates of Lemma 4.13 but treating the integrand as an analytic function of u and $\mathcal{I}(\nabla u)$. This yields a local Lipschitz estimate

$$\|\tilde{E}_2(u_1) - \tilde{E}_2(u_2)\|_{K^{l,\rho}} \lesssim \|u_1 - u_2\|_\rho + \|\nabla(u_1 - u_2)\|_\rho.$$

Now apply the Cauchy estimate for the second term; this shows that (CK3) is satisfied.

In a similar way, F_4 can be controlled by using the bounds on F_2, since F_2 appears in F_4.

The term E_3, which uses the auxillary operator used in (4.38) to define F_3 is defined by (4.36). It involves the following terms where ∇u appears,

$$\begin{aligned}
E_{31} &= \int_{\mathbb{T}} \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s, s_*)s_* \frac{h_s(s+s_*) - h_s(s)}{|\operatorname{Sin} s_*|^\beta} \,\mathrm{d}s_*,\\
E_{32} &= -2\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h_s)(\mathcal{I}(g)\cdot\mathcal{I}(h))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \,\mathrm{d}s_*,\\
E_{33} &= -\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h)(|\mathcal{I}(h)|^2 + \mathcal{I}(g)\cdot\mathcal{I}(h_s))s_*^3}{|\operatorname{Sin} s_*|^{\beta+2}} \,\mathrm{d}s_*,\\
E_{34} &= h_s G,
\end{aligned}$$

where G is a collection of terms involving only u and not ∇u, defined by the last four lines of (4.37). These terms are controlled by combining the above lemmas with the skew-symmetry (Lemma 4.7) and the Cauchy-type estimate of Lemma 4.15.

We now collect the above results and give the proof for the local existence and uniqueness of analytic sharp fronts.

Theorem 4.17. *Let $z_0 : \mathbb{T} \to \mathbb{R}^2$ be an analytic curve with $\Gamma_0 = \Gamma(z_0) \in L^\infty(\mathbb{T}^2)$. Then there exists $T^* > 0, \rho_0 > 0$ and $\beta > 0$ such that a unique solution to (4.1) exists in the space $u \in Y_{\rho_0,\beta,T^*}$.*

Proof. Since z_0 is analytic on $\mathbb{T}$, there exists $\rho > 0$ such that z_0 admits an analytic continuation to a complex neighbourhood $\mathbb{T} + \mathrm{i}(-\rho, \rho)$ of $\mathbb{T}$, also written z_0, that belongs to the space $K^{l,\rho}$ (recall that we have already fixed some $l \geq 2$). From z_0, we define the initial data to (4.43) as

$$u_0 = (z_0, \partial_s z_0, \partial_s^2 z_0, \Gamma(z_0)) \in X_{\rho_0}. \tag{4.47}$$

The operator F in (4.43) is continuous, satisfying (CK1) for Theorem 3.6, and there exists β_0 and T such that (CK2) is satisfied. With the estimates and earlier discussion in this subsection, (CK3) is satisfied, so Theorem 3.6 applies, proving the result. $\square$

Chapter 5

Almost-Sharp Fronts

In this chapter, we define an almost-sharp front in our setting that approximates a sharp front for a bounded domain. Then we will derive an asymptotic equation for an almost-sharp front, utilising an asymptotic expansion for a certain parameterised family of integrals. We also show that a function that was instrumental in the study of almost-sharp fronts of SQG has an analogue for our equation. It has a well-defined limit equation as $\delta \to 0$, which is better than the approximate equation for the almost-sharp front. This function, obtained by integrating across the transition region, simplifies the asymptotic equation of an almost-sharp front.

5.1 Almost-Sharp Fronts

We begin with the definition of an almost-sharp front.

Definition 5.1 (Almost-sharp front solution)**.** A δ almost-sharp front (ASF) $\theta = \theta_\delta(x,t)$ to singular SQG (1.1) is a family of solutions to to (1.1), such that for each $\delta > 0$ sufficiently small, there exists a closed C^2 curve z with regular parameterisation bounding a region A, and a constant $C^z > 0$ such that with the following sets,

$$\begin{aligned}
A_{\text{mid}}(t) = A^z_{\text{mid}}(t) &= \left\{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial A(t)) \leq C^z\delta\right\} = (\partial A)_\delta, \\
A_{\text{in}}(t) = A^z_{\text{in}}(t) &= \left\{x \in A : \operatorname{dist}(x, \partial A(t)) > C^z\delta\right\}, \\
A_{\text{out}}(t) = A^z_{\text{out}}(t) &= \mathbb{R}^2 \setminus (A_{\text{in}} \cup A_{\text{mid}}),
\end{aligned}$$

θ is a C^2 function such that

$$\theta(x,t) = \begin{cases} 1 & x \in A_{\text{in}}(t), \\ C^2 \text{ smooth} & x \in A_{\text{mid}}(t), \\ 0 & x \in A_{\text{out}}(t), \end{cases}$$

and for the family of solutions θ_δ, the following growth condition holds,

$$\|\nabla\theta\|_{L^\infty} \lesssim \frac{1}{\delta}. \tag{5.1}$$

In particular θ is locally constant in $A_{\text{out}}(t) \cup A_{\text{in}}(t)$, $\theta|_{A_{\text{in}}}(x,t) = 1$, $\theta|_{A_{\text{out}}}(x,t) = 0$, and the supports of θ and $\nabla\theta$ are related to the sets $A_{\text{mid}}, A_{\text{in}}, A_{\text{out}}$ by

$$\operatorname{supp}\theta \subset A_{\text{mid}}(t) \cup A_{\text{in}}(t), \quad \operatorname{supp}\nabla\theta(\cdot,t) \subset A_{\text{mid}}(t).$$

Note that the curve z in the above definition is not unique.

Definition 5.2 (Compatible curve)**.** Any curve z satisfying the above definition for an ASF θ is called a compatible curve for θ.

5.1.1 Tubular neighbourhood coordinates

Given a curve z with uniform speed parameterisation (as in Definition 1.16), we define the tubular neighbourhood coordinates around z by

$$x(s,\xi) = z(s) + \delta\xi N(s), \;\; s \in \mathbb{T}, \xi \in [-1,1]. \tag{5.2}$$

Suppose the curve also evolves in time $z = z(s,t)$. For convenience, we write $L_1 := L(1-\delta\kappa\xi)$. Setting $\tau = t$ as a new time variable, we have the Jacobian expressed as the block matrix(treating T, N as column vectors),

$$\frac{\mathrm{d}(x^1,x^2,t)}{\mathrm{d}(s,\xi,\tau)} = \left[\begin{array}{c|c} M_{2\times2} & x_\tau \\ \hline 0_{1\times2} & 1 \end{array}\right] = \left[\begin{array}{c|c|c} L_1 T & \delta N & x_\tau \\ \hline 0 & 0 & 1 \end{array}\right] \in \mathbb{R}^{3\times3},$$

with $x_\tau = z_\tau + \delta\xi N_\tau$, and determinant $L_1\delta$. This is a diffeomorphism if $L_1 = L(1-\delta\kappa\xi)$ is bounded away from 0, so in particular for $\delta \ll 1$. Its inverse is $\frac{\mathrm{d}(s,\xi,\tau)}{\mathrm{d}(x^1,x^2,t)} = \left[\begin{array}{c|c} M_{2\times2}^{-1} & -M_{2\times2}^{-1}x_\tau \\ \hline 0_{1\times2} & 1 \end{array}\right]$, as can be verified by direct calculation. Also

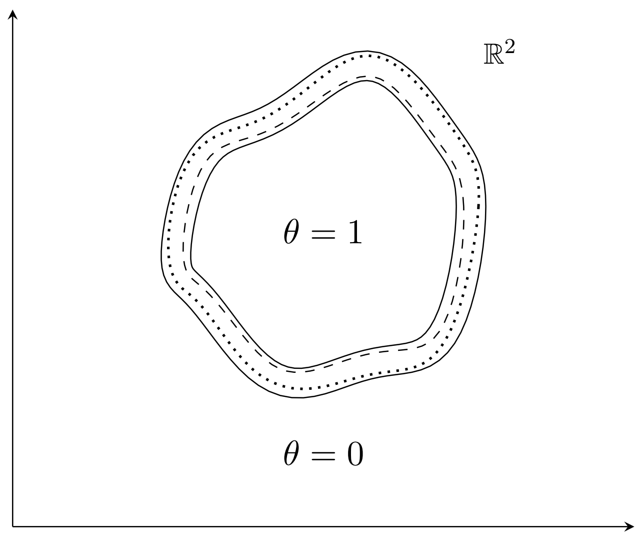

Figure 5.1: An illustration of an almost-sharp front for one $\delta > 0$. Here, the two solid curves form the boundary of A_{mid} (the compatible curve that defines A_{mid} has not been drawn); the dashed line is the boundary of $\{\theta = 1\}$, and the dotted line is the boundary of $\{\theta = 0\}$. In particular, the complement of $\{\theta = 0 \text{ or } 1\}$ is a subset of (and may not be equal to) A_{mid} .

define the column vector

$$A_0 = \begin{bmatrix} A_0^1 \\ A_0^2 \end{bmatrix} = -M_{2\times 2}^{-1} x_\tau = \begin{bmatrix} -x_\tau \cdot T/L_1 \\ -x_\tau \cdot N/\delta \end{bmatrix}.$$

Noticing that $N_\tau \cdot N = \frac{\partial_\tau}{2}|N|^2 = 0$, the $N_\tau \cdot N$ term that appears in A_0^2 vanishes. Therefore, the components are explicitly

$$\frac{\mathrm{d}(s,\xi,\tau)}{\mathrm{d}(x^1,x^2,t)} = \begin{bmatrix} \dfrac{T^1}{L_1} & \dfrac{T^2}{L_1} & A_0^1 \\ \dfrac{-T^2}{\delta} & \dfrac{T^1}{\delta} & A_0^2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} -\dfrac{1}{L_1} z_\tau \cdot T - \delta\xi N_\tau \cdot T \\ -\dfrac{1}{\delta} z_\tau \cdot N \end{bmatrix}.$$

Note that $L_1 = L + O(\delta)$, so $A_0^1 = \frac{-z_t \cdot T}{L} + O(\delta)$, and A_0^2 depends badly on δ. The derivatives in x^i, t are given in terms of the new coordinates s, ξ, τ as

$$\begin{aligned} \partial_{x^1} &= + \frac{T^1}{L_1}\partial_s - \frac{T^2}{\delta}\partial_\xi, \\ \partial_{x^2} &= + \frac{T^2}{L_1}\partial_s + \frac{T^1}{\delta}\partial_\xi, \\ \partial_t &= + A_0^1\partial_s + A_0^2\partial_\xi + \partial_\tau. \end{aligned}$$

and $\mathrm{d}x = \mathrm{d}x^1\,\mathrm{d}x^2 = \left|\det \frac{\mathrm{d}(x^1,x^2,t)}{\mathrm{d}(s,\xi,\tau)}\right| \mathrm{d}s\,\mathrm{d}\xi = L_1\delta\,\mathrm{d}s\,\mathrm{d}\xi$.

5.1.2 Equation in tubular neighbourhood coordinates

Define Ω to be θ expressed in the above tubular neighbourhood coordinates, $\Omega(s,\xi,\tau) = \theta(x,t)$. Then the gradient and its perpendicular can be written as

$$\nabla\theta = T\frac{\Omega_s}{L_1} + N\frac{\Omega_\xi}{\delta}, \quad \nabla^\perp\theta = N\frac{\Omega_s}{L_1} - T\frac{\Omega_\xi}{\delta}.$$

Recall that

$$u(x,t) = \int_{\mathbb{R}^2} \frac{\nabla^\perp\theta(x_*)}{|x - x_*|^{1+\alpha}}\,\mathrm{d}x_*.$$

So we have an expression of the velocity in new coordinates (using the notation that $x = x(s,\xi,\tau)$ and $x_* = x(s_*,\xi_*,\tau)$),

$$\begin{aligned} u(x) &= \iint_{\mathbb{T}\times[-1,1]} \frac{N_* \frac{\Omega_{s*}}{L_{1*}} - T_* \frac{\Omega_{\xi*}}{\delta}}{|x - x_*|^{1+\alpha}} \delta L_{1*} \, \mathrm{d}s_* \, \mathrm{d}\xi_* \\ &= \iint_{\mathbb{T}\times[-1,1]} \frac{\delta N_* \Omega_{s*} - L_{1*} T_* \Omega_{\xi*}}{|x - x_*|^{1+\alpha}} \, \mathrm{d}\xi_* \, \mathrm{d}s_*. \end{aligned}$$

Then since

$$\begin{aligned} \nabla\theta(x) \cdot \nabla^\perp\theta(x_*) = {} & N_* \cdot T \frac{\Omega_{s*}\Omega_s}{L_{1*}L_1} + N_* \cdot N \frac{\Omega_{s*}\Omega_\xi}{\delta L_{1*}} \\ & - T_* \cdot T \frac{\Omega_{\xi*}\Omega_s}{\delta L_1} - T_* \cdot N \frac{\Omega_{\xi*}\Omega_\xi}{\delta^2}, \end{aligned}$$

we can write the $u \cdot \nabla\theta$ term in the equation (1.1) as

$$\begin{aligned} u \cdot \nabla\theta(x) &= \iint_{\mathbb{T}\times[-1,1]} \frac{1}{|x - x_*|^{1+\alpha}} \big(N_* \cdot T \frac{\Omega_{s*}\Omega_s}{L_{1*}L_1} + N_* \cdot N \frac{\Omega_{s*}\Omega_\xi}{\delta L_{1*}} \\ &\quad - T_* \cdot T \frac{\Omega_{\xi*}\Omega_s}{\delta L_1} - T_* \cdot N \frac{\Omega_{\xi*}\Omega_\xi}{\delta^2} \big) \delta L_{1*} \, \mathrm{d}\xi_* \, \mathrm{d}s_* \\ &= \iint_{\mathbb{T}\times[-1,1]} \frac{1}{|x - x_*|^{1+\alpha}} \big(N_* \cdot T \frac{\delta\Omega_{s*}\Omega_s}{L_1} + N_* \cdot N \Omega_{s*}\Omega_\xi \\ &\quad - T_* \cdot T \frac{L_{1*}\Omega_{\xi*}\Omega_s}{L_1} - T_* \cdot N \frac{L_{1*}\Omega_{\xi*}\Omega_\xi}{\delta} \big) \, \mathrm{d}\xi_* \, \mathrm{d}s_*. \end{aligned}$$

Therefore, after using the identity

$$\frac{L_{1*}}{L_1} = 1 - \frac{\delta(\xi_*\kappa_* - \xi\kappa)}{1 - \delta\kappa\xi},$$

the equation (1.1) can be written in the new coordinates as (using $\nabla^\perp\Omega_* \cdot \nabla\Omega = \Omega_{s*}\Omega_\xi - \Omega_{\xi*}\Omega_s$ to collect terms)

$$\begin{aligned} 0 = {} & \Omega_s \left(A_0^1 + \iint_{\mathbb{T}\times[-1,1]} \frac{\delta N_* \cdot T}{L_1 |x_* - x|^{1+\alpha}} \Omega_{s*} \, \mathrm{d}\xi_* \, \mathrm{d}s_* \right. \\ & \left. + \iint_{\mathbb{T}\times[-1,1]} \frac{\delta(\xi_*\kappa_* - \xi\kappa) T_* \cdot T}{(1 - \delta\xi\kappa)|x_* - x|^{1+\alpha}} \Omega_{\xi*} \, \mathrm{d}\xi_* \, \mathrm{d}s_* \right) \\ & + \left(\iint_{\mathbb{T}\times[-1,1]} \frac{T_* \cdot T}{|x_* - x|^{1+\alpha}} \nabla^\perp\Omega_* \, \mathrm{d}\xi_* \, \mathrm{d}s_* \right) \cdot \nabla\Omega \\ & + \frac{1}{\delta}\Omega_\xi \left(\iint_{\mathbb{T}\times[-1,1]} \frac{-L_{1*} T_* \cdot N}{|x_* - x|^{1+\alpha}} \Omega_{\xi*} \, \mathrm{d}\xi_* \, \mathrm{d}s_* - z_\tau \cdot N \right) + \Omega_\tau. \end{aligned} \tag{5.3}$$

5.2 An approximate equation for an almost-sharp front

We introduce the symbols I_i to write the above of equation (5.3) as

$$0 = \Omega_s(A_0^1 + \delta I_1 + \delta I_2) + I_3 \cdot \nabla\Omega + \frac{\Omega_\xi}{\delta}(I_4 - z_\tau \cdot N), \tag{5.4}$$

where $A_0^1 = \frac{-z_\tau \cdot T}{L} + O(\delta)$ as mentioned before, and the other explicit dependencies on δ are visible, but there is still δ dependence in the integrals I_i through the coordinate function $x = z + \delta\xi N$ in (5.2).

We now want to find the leading order behaviour in $\delta \ll 1$. To do this, we will need to use the following two asymptotic results.

Lemma 5.3. *Let $\alpha \in (0,1)$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $s \in \mathbb{T}$, $\tau > 0$, let $I = I(\tau)$ denote the following family of integrals,*

$$I = \int_{s_* \in \mathbb{T}} \frac{a(s_*)}{|g(s_*) + \tau^2|^{(1+\alpha)/2}} \, \mathrm{d}s_*,$$

where $a = a(s_*), g = g(s_*) \in C^\infty(\mathbb{T})$ and $g = g(s_*)$ has 0 as its unique global minimum at the point $s_* = s$ and is non-degenerate, i.e.

$$g''(s) > 0, \quad \operatorname{argmin} g = s, \quad g(s) = \min g = 0.$$

Then we have the asymptotic expansion as $\tau \to 0$,

$$\begin{aligned} I = {} & \frac{a(s)}{G(s)} C_{1,\alpha} \tau^{-\alpha} + \frac{a(s) C_{2,\alpha}}{G(s)^{1+\alpha}} \\ & + \int_{\mathbb{T}} \frac{a(s_*)}{|g_*|^{(1+\alpha)/2}} - \frac{a(s)}{G(s)^{1+\alpha} |\operatorname{Sin}(s_* - s)|^{1+\alpha}} \, \mathrm{d}s_* + O(\tau^{2-\alpha}), \quad \tau \to 0, \end{aligned}$$

where:

1. $\operatorname{Sin} s_* := \sin(\pi s_*)/\pi$,
2. *C_α is the constant $C_\alpha = \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})} < \infty$ (which diverges as $\alpha \to 0$),*
3. *b_α is the constant $b_\alpha := \int_{-1/2}^{1/2} (\frac{1}{|s_*|^{1+\alpha}} - \frac{1}{|\operatorname{Sin} s_*|^{1+\alpha}}) \, \mathrm{d}s_* < \infty$,*
4. *G is the constant $G := \sqrt{g''(s)/2}$ which is well-defined since $g''(0) > 0$, and*
5. *the $O(\tau^{2-\alpha})$ constant depends on $W^{3,\infty}$ norms of a and g.*

The second result deals with the integral term that remains after applying Lemma 5.3.

Corollary 5.4. *Let $s \in \mathbb{T}$. For $\delta \in (-\delta_0, \delta_0)$ sufficiently small, let $H = H(\delta)$ denote the following family of integrals,*

$$H = \int_{s_* \in \mathbb{T}} \left(\frac{a(s_*)}{|g(s_*, \delta)|^{(1+\alpha)/2}} - \frac{a(s)}{G(\delta)^{1+\alpha} |\operatorname{Sin}(s_* - s)|^{1+\alpha}} \right) \mathrm{d}s_*,$$

where $a = a(s_*) \in C^\infty(\mathbb{T})$, , $g = g(s_*, \delta) \in C^\infty(\mathbb{T} \times [0, \infty))$, and g has a unique global minimum that is non-degenerate at $s_* = s$, i.e.

$$\operatorname{argmin} g(\cdot, \delta) = s, \quad g(s, \delta) = \min g(\cdot, \delta) = 0,$$

with $\partial_{s*}^2 g(0, \cdot) > c > 0$ for a constant c independent of δ, and $G(\delta) := \sqrt{\partial_{s*}^2 g(s, \delta)/2}$. Then we have the first order asymptotic expansion $H(\delta) = H(0) + H'(0)\delta + O(\delta^2)$ for $\delta \ll 1$, with

$$H'(0) = \int_{s_* \in \mathbb{T}} \left(\frac{a(s_*)\partial_\delta g(s_*, 0)(-1-\alpha)}{|g(s_*, 0)|^{(3+\alpha)/2}} - \frac{\partial_\delta (G^{-1-\alpha})(0) a(s)}{|\operatorname{Sin}(s - s_*)|^{1+\alpha}} \right) \mathrm{d}s_*.$$

We defer their proofs to the end of this chapter (as Lemma 5.8 and Corollary 5.9). To use these results, we will perform an approximation of the integral in s_* for fixed ξ, ξ_*, and s. We first sketch the proof in words, and then present the full calculation. Since we are interested in treating the denominator $|x - x_*|^{1+\alpha}$, we will need to use the above two results with the following choices:

$$\begin{aligned} \tau &= \delta(\xi_* - \xi), \\ g(s_*) &= g(s_*, \delta) = |x_* - x|^2 - \tau^2, \\ G(\delta) &= \sqrt{\frac{1}{2}\partial_{s*}^2 g(s, \delta)} = L(1 - \delta\kappa\xi_*). \end{aligned}$$

Note that if the numerator $a(s_*)$ vanishes at $s_* = s$, i.e. $a(s) = 0$, the terms in Lemma 5.3 and Corollary 5.4 simplify greatly. In addition, many terms in (5.4) are multiplied by δ, which simplifies the analysis of those terms for $\delta \ll 1$. We have:

1. For $I_1 = \iint_{\mathbb{T}\times[-1,1]} \frac{\delta N_* \cdot T}{L_1 |x_* - x|^{1+\alpha}} \Omega_{s*} \, \mathrm{d}\xi_* \, \mathrm{d}s_*$, it is multiplied by δ and $a(s) = 0$. This means that I_1 vanishes in the limit $\delta \to 0$.

2. For $I_2 = \iint_{\mathbb{T}\times[-1,1]} \frac{\delta(\xi_*\kappa_* - \xi\kappa) T_* \cdot T}{(1-\delta\xi\kappa)|x_* - x|^{1+\alpha}} \Omega_{\xi*} \, \mathrm{d}\xi_* \, \mathrm{d}s_*$, $a|_{s_*=s} \neq 0$ but it is multiplied by δ. This just gives us a $O(\delta^{1-\alpha})$ term instead, which is still $o(1)$.

3. For $I_4 = \iint_{\mathbb{T}\times[-1,1]} \frac{-L_{1*} T_* \cdot N}{|x_* - x|^{1+\alpha}} \Omega_{\xi*} \, \mathrm{d}\xi_* \, \mathrm{d}s_*$, $a|_{s_*=s} = 0$ but it is not multiplied by δ. I_4 has a δ correction term from the $L_{1*} = L(1 - \delta\kappa_*\xi_*)$ factor in the integrand, which becomes non-negligible after dividing by δ. Here, we see that there is no

hope for cancellation unless the curve used to create the coordinates evolves via the sharp front equation (at least, up to $O(\delta)$ errors). Therefore, we are forced to impose that z evolves by the sharp front equation. Then, the identity $\int_{-1}^{1} \partial_\xi \Omega = 1$, Lemma 5.3 , and Corollary 5.4 gives us that:

$$
\begin{aligned}
&\frac{1}{\delta}(I_4 - z_\tau \cdot N) \\
&= (2+2\alpha) \int_{-1}^{1} \int_{\mathbb{T}} \frac{LT_* \cdot N}{|z - z_*|^{3+\alpha}} (z_* - z) \cdot (\xi_* N_* - \xi N) \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* \\
&\quad + \int_{-1}^{1} \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N}{|z - z_*|^{1+\alpha}} \xi_* \partial_\xi \Omega_* \, \mathrm{d}s_{**} \, \mathrm{d}\xi_* \Omega_\xi + o(1).
\end{aligned}
$$

4. For $I_3 = \iint_{\mathbb{T}\times[-1,1]} \frac{T_* \cdot T}{|x_* - x|^{1+\alpha}} \nabla^\perp \Omega_* \, \mathrm{d}\xi_* \, \mathrm{d}s_*$, there is no positive power of δ and $a|_{s_*=s} \neq 0$. This term cannot be dealt with in the same way and contains a term that is divergent as $\delta \to 0$. In the notation of Lemma 5.3, we have $a(s) = T(s) \cdot T(s_*)$ i.e. $a = T \cdot T_*$ with $a(0) = 1$. To apply Corollary 5.4, we note that G at $\delta = 0$ is L. Thus, I_3 can be written as follows,

$$
\begin{aligned}
I_3 &= \iint_{\mathbb{T}\times[-1,1]} \frac{T_* \cdot T}{|x - x_*|^{1+\alpha}} \nabla \Omega_*^\perp \, \mathrm{d}\xi_* \, \mathrm{d}s_* \cdot \nabla \Omega \\
&= \frac{C_{1,\alpha}}{L\delta^\alpha} \int_{-1}^{1} \frac{\nabla^\perp \Omega_*|_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* + \frac{C_{2,\alpha}}{L^{1+\alpha}} \int_{-1}^{1} \nabla^\perp \Omega_*|_{s_*=s} \, \mathrm{d}\xi \\
&\quad + \int_{\mathbb{T}} \frac{T_* \cdot T \int_{-1}^{1} \nabla^\perp \Omega_* \, \mathrm{d}\xi_*}{|z - z_*|^{1+\alpha}} - \frac{\int_{-1}^{1} \nabla^\perp \Omega_*|_{s_*=s} \, \mathrm{d}\xi_*}{L^{1+\alpha} |\operatorname{Sin}(s - s_*)|^{1+\alpha}} \, \mathrm{d}s_* + o(1).
\end{aligned}
$$

Hence, the equation is of the approximate form $o(1) = \partial_\tau \Omega + X \cdot \nabla \Omega + I_3 \cdot \nabla \Omega$, which we write as the following theorem.

Theorem 5.5 (Approximate Equation for an ASF)**.** *Ω is a δ-ASF for singular SQG (1.1) in the sense of Definition 5.1 iff in the tubular neighbourhood of the sharp front z, it solves the following approximate equation,*

$$
\begin{aligned}
o(1) = \partial_\tau \Omega &- \frac{z_\tau \cdot T}{L} \partial_s \Omega \\
&+ (2+2\alpha)L \int_{-1}^{1} \int_{\mathbb{T}} \frac{T_* \cdot N}{|z - z_*|^{3+\alpha}} (z - z_*) \cdot (\xi N - \xi_* N_*) \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \int_{-1}^{1} \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N}{|z - z_*|^{1+\alpha}} \xi_* \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \frac{C_{1,\alpha}\delta^{-\alpha}}{L} \int_{-1}^{1} \frac{\nabla \Omega_*^\perp|_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* \cdot \nabla \Omega
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C_{2,\alpha}}{L^{1+\alpha}}\int_{-1}^{1}\nabla\Omega^{\perp}|_{s_*=s}\,\mathrm{d}\xi_*\cdot\nabla\Omega \\
&+\int_{-1}^{1}\int_{\mathbb{T}}\frac{T_*\cdot T\nabla^{\perp}\Omega_*\cdot\nabla\Omega}{|z-z_*|^{(1+\alpha)/2}}-\frac{\nabla^{\perp}\Omega_*|_{s_*=s}\cdot\nabla\Omega}{L^{1+\alpha}|\operatorname{Sin}(s-s_*)|^{1+\alpha}}\,\mathrm{d}s_*\,\mathrm{d}\xi_*. \qquad (5.5)
\end{aligned}
$$

Remark 5.6. In (5.5), the integral term

$$
2(1+\alpha)L\int_{-1}^{1}\int_{\mathbb{T}}\frac{T_*\cdot N}{|z-z_*|^{3+\alpha}}(z-z_*)\cdot(\xi N-\xi_*N_*)\partial_\xi\Omega_*\,\mathrm{d}s_*\,\mathrm{d}\xi_*\partial_\xi\Omega
$$

is well-defined, despite the mismatch $\xi\neq\xi_*$, which makes it seem like the singularity in the denominator is not compensated for. This is because the dot product introduces a second cancellation as long as say, $z\in C^3(\mathbb{T})$. One can see this by applying Taylor's theorem to z_* and N_*,

$$
\begin{aligned}
z_*-z&=T(s-s_*)+O((s-s_*)^2),\\
N_*&=N-L\kappa T(s-s_*)+O((s-s_*)^2),
\end{aligned}
$$

which implies that

$$
\xi N-\xi_*N_*=(\xi-\xi_*)N+\xi_*L\kappa T(s-s_*)+O((s-s_*)^2),
$$

so that

$$
(z_*-z)\cdot(\xi N-\xi_*N_*)=\xi_*L\kappa(s-s_*)^2+O((s-s_*)^3).
$$

Proof of Theorem 5.5. Here we record the computations summarised above. Recall that s, ξ, and ξ_* are fixed when applying Lemma 5.3, so we will write $a_*=a(s_*)=a(s,s_*,\xi,\xi_*)$, and $a=a(s)=a(s,s,\xi,\xi_*)$ (i.e. $a(s_*)$ at the value $s_*=s$).

Note that we will use slightly different notation from Lemma 5.8, since we have integrals in s_* instead of s. In addition, our unique non-degenerate minimiser of the function $g_*=g(s_*)$ is at $s_*=s$, (for every $\delta>0$ sufficiently small) as we will show below. More specifically, we have

$$
\begin{aligned}
x&=z+\xi\delta N,\\
x_*&=z+\xi_*\delta N_*,\\
x_{s*}=\partial_{s_*}x_*&=L(1-\xi_*\delta\kappa_*)T_*,\\
|x_*-x|^2&=|x_*|^2+|x|^2-2x_*\cdot x,\\
\tau^2&=\delta^2(\xi_*-\xi)^2,\\
g(s_*)&=|x_*-x|^2-\tau^2,
\end{aligned}
$$

$$
\begin{aligned}
g'(s_*) &= 2x_{s*} \cdot x_* - 2x_{s*} \cdot x \\
&= 2x_{s*} \cdot (x_* - x), \\
g''(s_*) &= 2|x_{s*}|^2 + 2x_{ss*} \cdot (x_* - x) \\
&= 2L^2(1 - \xi_* \delta \kappa_*)^2 + 2x_{ss*} \cdot (x_* - x).
\end{aligned}
$$

Since (as a function of s_* with s, ξ, ξ_* fixed) $g'(s) = 0$ and $g''(s) > 0$ for $\delta \ll 1$, g has a non-degenerate minimum at $s_* = s$, with $g(s) = \tau^2 - \tau^2 = 0$. As the curve z has no self-intersections, this is the unique minimum. Therefore we have (as in Lemma 5.3)

$$
G(\delta) = \sqrt{\frac{1}{2} g''(s)} = L(1 - \delta \kappa \xi_*).
$$

(Note that $G \neq L_1$: as G involves derivatives in s_* of g evaluated at $s_* = s$, ξ_* appears in G but not ξ.) For numerators $a_* = a(s_*)$, Lemma 5.3 gives $I = \int_{\mathbb{T}} \frac{a_*}{|g_* + \tau^2|^{(1+\alpha)/2}} \, \mathrm{d}s_*$ the expansion,

$$
\begin{aligned}
I = {} & \frac{a(s)}{G} C_{1,\alpha} \tau^{-\alpha} + \frac{a(s) C_{2,\alpha}}{G^{1+\alpha}} \\
& + \int_{\mathbb{T}} \frac{a_*}{|g_*|^{(1+\alpha)/2}} - \frac{a(s)}{G^{1+\alpha} |\operatorname{Sin}(s_* - s)|^{1+\alpha}} \, \mathrm{d}s_* + O(\tau^{2-\alpha}), \qquad \tau \to 0,
\end{aligned}
$$

where $\operatorname{Sin}(s_* - s) := \sin(\pi(s_* - s))/\pi$, and $C_{1,\alpha}, C_{2,\alpha}$ are two known constants. Also, with $g_* = g(s_*, \delta), G = G(\delta)$,

$$
\begin{aligned}
\partial_\delta g(s_*) &= 2\partial_\delta (x_* - x) \cdot (x_* - x) - 2\delta(\xi_* - \xi) \\
&= 2(\xi_* N_* - \xi N) \cdot (z_* - z) + 2\delta |\xi_* N_* - \xi N|^2 - 2\delta |\xi_* - \xi|^2, \\
\partial_\delta G &= -L \kappa \xi_*,
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial_\delta g(s_*, 0) &= 2(\xi_* N_* - \xi N) \cdot (z_* - z), \\
\partial_\delta (G^{-1-\alpha})(\delta) &= (-1 - \alpha) G^{-2-\alpha} \partial_\delta G = \frac{(1 + \alpha)}{L^{1+\alpha}(1 - \xi_* \delta \kappa)^{2+\alpha}} \kappa \xi_*.
\end{aligned}
$$

Corollary 5.4 in this setting gives

$$
\begin{aligned}
I = {} & \frac{a(s)}{L} C_{1,\alpha} \delta^{-\alpha} |\xi_* - \xi|^{-\alpha} + \frac{a(s) C_{2,\alpha}}{L^{1+\alpha}} \\
& + \int_{\mathbb{T}} \frac{a(s_*)}{|g(s_*, 0)|^{(1+\alpha)/2}} - \frac{a(s)}{L^{1+\alpha} |\operatorname{Sin}(s_* - s)|^{1+\alpha}} \, \mathrm{d}s_*
\end{aligned}
$$

$$
+\,\delta\int_{\mathbb{T}}\frac{(-2-2\alpha)a(s)}{|g(s_*,0)|^{(3+\alpha)/2}}(\xi_*N_*-\xi N)\cdot(z_*-z)-\frac{\partial_\delta(G^{-1-\alpha})(0)a(s)}{|\operatorname{Sin}s|^{1+\alpha}}\,\mathrm{d}s_*
$$
$$
+\,O(\delta^{1-\alpha}).
$$

We now detail the calculations that treat the integral terms I_i in (5.4). The first two terms I_1, I_2 are simpler, so we will deal with them first. The exact form of a and g is not so important once we know the behaviour as $\delta\to 0$. For I_1, $a(s_*)=\frac{N_*\cdot T}{L(1-\delta\kappa\xi)}\Omega_{s*}$, with $a(s)=0$. Thus, Lemma 5.3 gives

$$
\delta I_1=\delta\int_{-1}^{1}\int_{\mathbb{T}}\frac{a_*}{|g_*|^{(1+\alpha)/2}}\,\mathrm{d}s_*\,\mathrm{d}\xi_*+O(\delta^{2-\alpha})=O(\delta)=o(1).
$$

For I_2, $a(s_*)=\frac{(\xi_*\kappa_*-\xi\kappa)T_*\cdot T}{1-\delta\xi\kappa}\Omega_{\xi*}$. Thus, Lemma 5.3 gives

$$
\begin{aligned}
\delta I_2&=\delta^{1-\alpha}\int_{-1}^{1}|\xi-\xi_*|^{-\alpha}\frac{a}{L}C_{1,\alpha}\,\mathrm{d}\xi_*+\delta\frac{aC_{2,\alpha}}{L^{1+\alpha}}\\
&\quad+\delta\int_{\mathbb{T}}\frac{a_*}{|g_*|^{(1+\alpha)/2}}-\frac{\pi^{1+\alpha}a}{L^{1+\alpha}|\sin(\pi(s_*-s))|^{1+\alpha}}\,\mathrm{d}s_*+O(\delta^{1-\alpha})\\
&=O(\delta^{1-\alpha})=o(1).
\end{aligned}
$$

For I_4, since $L_{1*}=L(1-\delta\kappa_*\xi_*)$, we write $I_4=I_{4,1}+\delta I_{4,2}$, with

$$
\begin{aligned}
I_{4,1}&=\int_{-1}^{1}\int_{\mathbb{T}}\frac{-LT_*\cdot N}{|x_*-x|^{1+\alpha}}\Omega_{\xi*}\ \mathrm{d}\xi_*\,\mathrm{d}s_*,\\
I_{4,2}&=\int_{-1}^{1}\int_{\mathbb{T}}\frac{L\kappa_*\xi_*T_*\cdot N}{|x_*-x|^{1+\alpha}}\Omega_{\xi*}\ \mathrm{d}\xi_*\,\mathrm{d}s_*.
\end{aligned}
$$

For $I_{4,1}$, $a(s_*)=-LT_*\cdot N\Omega_{\xi*}$, and $a(s)=0$. Therefore, Lemma 5.3 gives

$$
\begin{aligned}
I_{4,1}&=\int_{-1}^{1}\int_{\mathbb{T}}\frac{a_*}{|g_*|^{(1+\alpha)/2}}\,\mathrm{d}s_*\,\mathrm{d}\xi_*+O(\delta^{2-\alpha})\\
&=-L\int_{-1}^{1}\int_{\mathbb{T}}\frac{T_*\cdot N\Omega_{\xi*}}{|g_*|^{(1+\alpha)/2}}\,\mathrm{d}s_*\,\mathrm{d}\xi_*+O(\delta^{2-\alpha}),
\end{aligned}
$$

and Corollary 5.4 gives (since $\partial_\delta g(s_*,0)=L(x-x_*)\cdot(\xi N-\xi_*N_*)$)

$$
\begin{aligned}
I_{4,1}=-L&\int_{\mathbb{T}}\frac{T_*\cdot N\int_{-1}^{1}\Omega_{\xi*}\,\mathrm{d}\xi_*}{|z-z_*|^{1+\alpha}}\,\mathrm{d}s_*+O(\delta^{2-\alpha})\\
&+(1+\alpha)\delta\int_{-1}^{1}\int_{\mathbb{T}}\frac{2LT_*\cdot N}{|z-z_*|^{3+\alpha}}(z-z_*)\cdot(\xi N-\xi_*N_*)\partial_\xi\Omega_*\,\mathrm{d}s_*\,\mathrm{d}\xi_*\\
&+O(\delta^{2-\alpha}).
\end{aligned}
$$

Since we have chosen $\int_{-1}^{1} \Omega_{\xi*}\, \mathrm{d}\xi_* = 1$, the first term is equal to the sharp front evolution term $-\mathcal{I}(z) \cdot N = z_\tau \cdot N$, and

$$\begin{aligned}
&\frac{1}{\delta}(I_{4,1} - z_\tau \cdot N) = \\
&= (1+\alpha) \int_{-1}^{1} \int_{\mathbb{T}} \frac{2LT_* \cdot N}{|z - z_*|^{3+\alpha}} (z - z_*) \cdot (\xi N - \xi_* N_*) \partial_\xi \Omega_* \, \mathrm{d}s_* \, \mathrm{d}\xi_* + o(1).
\end{aligned}$$

For $I_{4,2}$, $a(s_*) = L\kappa_* \xi_* T_* \cdot N \Omega_{\xi*}$, and Lemma 5.3 and Corollary 5.4 gives

$$\begin{aligned}
I_{4,2} &= \int_{-1}^{1} \int_{\mathbb{T}} \frac{L\kappa_* \xi_* T_* \cdot N \Omega_{\xi*}}{|x_* - x|^{1+\alpha}} \, \mathrm{d}\xi_* \, \mathrm{d}s_* \\
&= \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N \int_{-1}^{1} \xi_* \Omega_{\xi*} \, \mathrm{d}\xi_*}{|z_* - z|^{1+\alpha}} \, \mathrm{d}s_* + O(\delta).
\end{aligned}$$

For I_3, $a(s_*) = T_* \cdot T \nabla^\perp \Omega_*$. Applying Lemma 5.3 gives

$$\begin{aligned}
I_3 = \int_{-1}^{1} \Bigg(& \frac{C_{1,\alpha}}{L(1 - \delta\kappa\xi_*)\delta^\alpha} \frac{\nabla^\perp \Omega_*|_{s_*=s}}{|\xi - \xi_*|^\alpha} + \frac{C_{2,\alpha}}{(L(1 - \delta\kappa\xi_*))^{1+\alpha}} \nabla^\perp \Omega_*|_{s_*=s} \\
&+ \int_{\mathbb{T}} \frac{T_* \cdot T \nabla^\perp \Omega_*}{|g_*|^{(1+\alpha)/2}} - \frac{\pi^{1+\alpha} \nabla^\perp \Omega_*|_{s_*=s}}{(L(1 - \delta\kappa\xi_*))^{1+\alpha} |\sin(\pi(s - s_*))|^{1+\alpha}} \, \mathrm{d}s_* \Bigg) \mathrm{d}\xi_* \\
&+ O(\delta^{1-\alpha}).
\end{aligned}$$

The remaining integral in s_* is dealt with by applying Corollary 5.4,

$$\begin{aligned}
I_3 = {} & \frac{C_{1,\alpha}}{L\delta^\alpha} \int_{-1}^{1} \frac{\nabla^\perp \Omega_*|_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* + \frac{C_{2,\alpha}}{L^{1+\alpha}} \int_{-1}^{1} \nabla^\perp \Omega_*|_{s_*=s} \, \mathrm{d}\xi_* \\
&+ \int_{\mathbb{T}} \frac{T_* \cdot T \int_{-1}^{1} \nabla^\perp \Omega_* \, \mathrm{d}\xi_*}{|z - z_*|^{1+\alpha}} - \frac{\pi^{1+\alpha} \int_{-1}^{1} \nabla^\perp \Omega_*|_{s_*=s} \, \mathrm{d}\xi_*}{L^{1+\alpha} |\sin(\pi(s - s_*))|^{1+\alpha}} \, \mathrm{d}s_* + O(\delta^{1-\alpha}).
\end{aligned}$$

This covers all the integral terms in the equation, which completes the proof. $\square$

Remark 5.7. As opposed to the SQG case which corresponds to $\alpha = 0$, there is a 'single bad term' with respect to δ and a kernel in ξ (without s) appearing. For $\alpha = 0$, this term is split into two terms, a bad $O(\log \delta)$ term (removable by unwinding the singularity, see [28]) and a logarithmic kernel term (see Remark 5.10, and also [28]). The analogous derivation as above for an almost-sharp front in the sense of our definition leads to the following approximate SQG equation:

$$o(1) = \partial_\tau \Omega - \frac{z_\tau \cdot T}{L} \partial_s \Omega$$

$$
\begin{aligned}
&+ 2L \int_{-1}^{1} \int_{\mathbb{T}} \frac{T_* \cdot N}{|z - z_*|^3} (z - z_*) \cdot (\xi N - \xi_* N_*) \partial_\xi \Omega_* \,\mathrm{d}s_* \,\mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \int_{-1}^{1} \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N}{|z - z_*|} \xi_* \partial_\xi \Omega_* \,\mathrm{d}s_* \,\mathrm{d}\xi_* \partial_\xi \Omega \\
&+ \frac{C_{1,0}}{L} \int_{-1}^{1} \nabla \Omega_*^{\perp}|_{s_*=s} \log |\xi - \xi_*| \,\mathrm{d}\xi_* \cdot \nabla \Omega \\
&+ \frac{C_{2,0} + C_{1,0} \log \delta}{L} \int_{-1}^{1} \nabla \Omega^{\perp}|_{s_*=s} \,\mathrm{d}\xi_* \cdot \nabla \Omega \\
&+ \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{T_* \cdot T \nabla \Omega_*^{\perp}}{|z - z_*|} - \frac{\pi \nabla \Omega_*^{\perp}|_{s_*=s}}{L |\sin(\pi(s - s_*))|} \right) \mathrm{d}s_* \,\mathrm{d}\xi_* \cdot \nabla \Omega.
\end{aligned}
$$

5.3 A regularisation by integration across the almost-sharp front

As noticed in [28], the terms involving $\nabla\Omega^{\perp} \cdot \nabla\Omega_*|_{s_*=s}$ in the above equation disappear on integration in ξ over $[-1,1]$, since for any even integrable function F and two functions f, g, we have the cancellation

$$
\iint_{\xi,\xi_* \in [-1,1]} F(\xi - \xi_*) \big[f(\xi) g(\xi_*) - f(\xi_*) g(\xi) \big] \,\mathrm{d}\xi \,\mathrm{d}\xi_* = 0,
$$

and the integral terms in (5.5) with $\nabla\Omega^{\perp} \cdot \nabla\Omega_*|_{s_*=s}$ are of this form with $f = \partial_s \Omega$ and $g = \partial_\xi \Omega$, for every s fixed. In particular, the term of order $\delta^{-\alpha}$ can be written in this form with $f = \partial_s \Omega$ and $g = \partial_\xi \Omega$. This motivates shifting Ω by a constant so that (keeping in mind that $\xi = 1$ corresponds to the point $z + \delta N$ and N is the inward normal)

$$
\Omega(s, \xi, t) = \begin{cases} -\frac{1}{2} & \xi \le -1, \\ C^2 \text{ smooth} & \xi \in [-1, 1], \\ \frac{1}{2} & \xi \ge 1, \end{cases} \tag{5.6}
$$

and we make the definition

$$
h(s) := \int_{-1}^{1} \Omega \,\mathrm{d}\xi.
$$

Then we have the identities

$$
\begin{aligned}
\int_{-1}^{1} \xi \partial_\xi \Omega(s, \xi) \,\mathrm{d}\xi &= -h(s), \\
\int_{-1}^{1} \partial_\xi \Omega(s, \xi) \,\mathrm{d}\xi &= 1.
\end{aligned}
$$

(This follows from direct computation, e.g. $\int_{-1}^{1} \xi \partial_\xi \Omega(s,\xi)\,\mathrm{d}\xi = \int_{-1}^{1} \xi \partial_\xi \Omega(s,\xi)\,\mathrm{d}\xi = \xi\Omega(s,\xi)|_{\xi=-1}^{1} - \int_{-1}^{1} \Omega(s,\xi)\,\mathrm{d}\xi = -h(s)$.) Thus we discover that $h = h(s)$ satisfies a much better behaved approximate equation, obtained by integrating the equation for Ω (5.5),

$$
\begin{aligned}
o(1) = \partial_\tau h &- \frac{z_\tau \cdot T}{L} h' \\
&+ (2+2\alpha)L \int_{s_*} \frac{T_* \cdot N}{|z-z_*|^{3+\alpha}} (z-z_*) \cdot (hN - h_* N_*)\,\mathrm{d}s_* \\
&+ \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N}{|z-z_*|^{1+\alpha}} h_*\,\mathrm{d}s_* \\
&+ \int_{\mathbb{T}} \left(\frac{T_* \cdot T\binom{-1}{h'_*}}{|z-z_*|^{1+\alpha}} - \frac{\pi^{1+\alpha}\binom{-1}{h'}}{L^{1-\alpha}|\sin(\pi(s-s_*))|^{1+\alpha}} \right) \mathrm{d}s_* \cdot \begin{pmatrix} h' \\ 1 \end{pmatrix}.
\end{aligned}
$$

Since $\binom{-1}{h'} = \binom{h'}{1}^{\perp}$, we can rewrite the last integral term to get the equation

$$
\begin{aligned}
o(1) = \partial_\tau h &- \frac{z_\tau \cdot T}{L} h' \\
&+ (2+2\alpha)L \int_{s_*} \frac{T_* \cdot N}{|z-z_*|^{3+\alpha}} (z-z_*) \cdot (hN - h_* N_*)\,\mathrm{d}s_* \\
&+ \int_{\mathbb{T}} \frac{L\kappa_* T_* \cdot N}{|z-z_*|^{1+\alpha}} h_*\,\mathrm{d}s_* + \int_{\mathbb{T}} \frac{T_* \cdot T(h'_* - h')}{|z-z_*|^{1+\alpha}}\,\mathrm{d}s_*.
\end{aligned}
$$

Namely, in the limit $\delta \to 0$, h evolves via a linear homogenous integrodifferential equation that does no t depend on Ω. so by rewriting the ξ_* integrals in (5.5) using the two further identities

$$
\int_{-1}^{1} (\xi N - \xi_* N_*) \partial_\xi \Omega_*\,\mathrm{d}\xi = (\xi N + h_* N_*), \qquad \int_{-1}^{1} \nabla \Omega^{\perp}|_{s_*=s}\,\mathrm{d}\xi_* = \begin{pmatrix} -1 \\ h' \end{pmatrix},
$$

we can treat h as an independently evolving function coupled with Ω via the following equation,

$$
\begin{aligned}
o(1) = {} & \partial_\tau \Omega - \frac{z_\tau \cdot T}{L} \partial_s \Omega \\
& + (2+2\alpha) L \int_{\mathbb{T}} \frac{T_* \cdot N}{|z - z_*|^{3+\alpha}} (z - z_*) \cdot (\xi N + h_* N_*) \, \mathrm{d}s_* \partial_\xi \Omega \\
& + \int_{\mathbb{T}} \frac{L \kappa_* T_* \cdot N}{|z - z_*|^{1+\alpha}} h_* \, \mathrm{d}s_* \partial_\xi \Omega \\
& + \frac{C_{1,\alpha} \delta^{-\alpha}}{L} \int_{-1}^{1} \frac{\nabla \Omega_*^\perp |_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* \cdot \nabla \Omega \\
& + \frac{C_{2,\alpha}}{L^{1+\alpha}} \begin{pmatrix} -1 \\ h' \end{pmatrix} \cdot \nabla \Omega \\
& + \int_{\mathbb{T}} \left(\frac{T_* \cdot T \binom{-1}{h'_*}}{|z - z_*|^{1+\alpha}} - \frac{\binom{-1}{h'}}{L^{1-\alpha} |\operatorname{Sin}(s - s_*)|^{1+\alpha}} \right) \mathrm{d}s_* \cdot \nabla \Omega.
\end{aligned}
$$

Thus, the introduction of h reduces the understanding of the evolution of Ω to understanding a nonlinear system, where the main nonlinearity is in the single term $L^{-1} C_{1,\alpha} \delta^{-\alpha} \int_{-1}^{1} \frac{\nabla \Omega_*^\perp |_{s_*=s}}{|\xi - \xi_*|^\alpha} \, \mathrm{d}\xi_* \cdot \nabla \Omega$.

In the case of the SQG equation, this term plays a central role in choosing a reparameterisation to prove existence of almost-sharp fronts for SQG for a uniform time independent of δ. (See [30] for more details.)

5.4 Asymptotics for a parameterised integral

In this section, we prove Lemma 5.3 and Corollary 5.4 that were used in proving Theorem 5.5. We will state and prove a slightly simpler but equivalent version, which comes from the following change of variables: recall from the notation of Lemma 5.3 that s is the minimiser of g, and g is a function of s_*. Set

$$
\begin{aligned}
\tilde{s} &= s_* - s, \\
\tilde{a}(\tilde{s}) &= a(s + \tilde{s}), \\
\tilde{g}(\tilde{s}) &= g(s + \tilde{s}).
\end{aligned}
$$

The minimiser of $\tilde{g}$ is $\tilde{s} = 0$. Dropping the tildes, we arrive at the following lemma, which we now prove.

Lemma 5.8. *Let $\alpha \in (0,1)$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $\tau > 0$, let $I = I(\tau)$ denote the following family of integrals,*

$$
I = \int_{s \in \mathbb{T}} \frac{a(s)}{|g(s) + \tau^2|^{(1+\alpha)/2}} \, \mathrm{d}s,
$$

where $a = a(s), g = g(s) \in C^\infty(\mathbb{T})$ and g has 0 as its unique global minimum at $s = 0$ that is non-degenerate, i.e.

$$g''(0) > 0, \quad \operatorname{argmin} g = 0, \quad g(0) = \min g = 0.$$

Then we have the asymptotic expansion as $\tau \to 0$,

$$\begin{aligned} I = {} & \frac{a(0)}{G} C_\alpha \tau^{-\alpha} - \frac{a(0)(\alpha^{-1}2^{1+\alpha} + b_\alpha)}{G^{1+\alpha}} \\ & + \int_{s \in \mathbb{T}} \frac{a(s)}{|g(s)|^{(1+\alpha)/2}} - \frac{a(0)}{G^{1+\alpha} |\operatorname{Sin} s|^{1+\alpha}} \,\mathrm{d}s + O(\tau^{2-\alpha}), \quad (\tau \to 0), \end{aligned}$$

where:

1. $\operatorname{Sin} s := \sin(\pi s)/\pi$,
2. *C_α is the constant $C_\alpha = \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})} < \infty$ (which diverges as $\alpha \to 0$),*
3. *b_α is the constant $b_\alpha := \int_{-1/2}^{1/2} (\frac{1}{|s|^{1+\alpha}} - \frac{1}{|\operatorname{Sin} s|^{1+\alpha}}) \,\mathrm{d}s < \infty$,*
4. *G is the constant $G := \sqrt{g''(0)/2}$ which is well-defined since $g''(0) > 0$, and*
5. *the $O(\tau^{2-\alpha})$ constant depends on $W^{3,\infty}$ norms of a and g.*

Proof. The proof will roughly follow the structure of the proof of the auxillary lemma of [28] that corresponds to the case $\alpha = 0$, except in parts where the fact that $\alpha > 0$ is needed, e.g. $C_\alpha < \infty$. Here, we take $(-1/2, 1/2)$ as a fundamental domain for $\mathbb{T}$. We split $I = I_{\text{near}} + I_{\text{far}}$ into an integral I_{near} around the minimiser of g and I_{far} on the complement,

$$\begin{aligned} A_{\text{near}} &:= (s_-, s_+), & I_{\text{near}} &:= \int_{A_{\text{near}}} \frac{a(s)}{|g(s) + \tau^2|^{(1+\alpha)/2}} \,\mathrm{d}s, \\ A_{\text{far}} &:= \mathbb{T} \setminus A_{\text{near}}, & I_{\text{far}} &:= \int_{A_{\text{far}}} \frac{a(s)}{|g(s) + \tau^2|^{(1+\alpha)/2}} \,\mathrm{d}s, \end{aligned} \tag{5.7}$$

where $s_\pm$ are chosen (depending on g) sufficiently close to 0 so that we can choose new coordinates σ such that $g(s) = \sigma^2$, and that $g(s_-) = g(s_+) =: \sigma_0^2 \ll 1$. Also define $\bar{a}(\sigma)$ so that $\bar{a}(\sigma) \,\mathrm{d}\sigma = a(s) \,\mathrm{d}s$ in the integral. Thus $I_{\text{near}} = \int_{-\sigma_0}^{\sigma_0} \frac{\bar{a}(\sigma)}{|\sigma^2 + \tau^2|^{(1+\alpha)/2}} \,\mathrm{d}\sigma$, so this reduces understanding I_{near} into understanding this change of coordinates, and an integral with a simpler denominator. We decompose I_{near} further into 3 parts:

$$\begin{aligned}
I_{\text{near}} &= \int_{-\sigma_0}^{\sigma_0} \frac{\bar{a}(\sigma)}{|\sigma^2+\tau^2|^{(1+\alpha)/2}}\,\mathrm{d}\sigma = I_{\text{near},1} + I_{\text{near},2} + I_{\text{near},3},\\
I_{\text{near},1} &= \bar{a}(0)\int_{-\sigma_0}^{\sigma_0} \frac{1}{|\sigma^2+\tau^2|^{(1+\alpha)/2}}\,\mathrm{d}\sigma,\\
I_{\text{near},2} &= \int_{-\sigma_0}^{\sigma_0} \frac{\bar{a}(\sigma)-\bar{a}(0)}{|\sigma|^{1+\alpha}}\,\mathrm{d}\sigma,\\
I_{\text{near},3} &= \int_{-\sigma_0}^{\sigma_0} \big(\bar{a}(\sigma)-\bar{a}(0)-\bar{a}'(0)\sigma\big)\left(\frac{1}{|\sigma^2+\tau^2|^{(1+\alpha)/2}} - \frac{1}{|\sigma|^{1+\alpha}}\right)\mathrm{d}\sigma.
\end{aligned}$$

By the regularity of $\bar{a}$, these integrals are well-defined. $I_{\text{near},1}$ is the only term that appears for a constant function $\bar{a} \equiv \bar{a}(0)$. $I_{\text{near},2}$ is bounded independent of τ. $I_{\text{near},3}$ can easily be seen to be $O(\tau^{2-\alpha})$ using the following simple bound,

$$\left|\frac{1}{|\sigma^2+\tau^2|^{(1+\alpha)/2}} - \frac{1}{\sigma^{1+\alpha}}\right| \le \begin{cases} 2\sigma^{-1-\alpha} & |\sigma| \le \tau,\\ \tau^2\sigma^{-3-\alpha} & |\sigma| > \tau. \end{cases}$$

The first bound follows from the triangle inequality and $\frac{1}{|\sigma^2+\tau^2|^{(1+\alpha)/2}} \le \frac{1}{|\sigma|^{1+\alpha}}$; the second bound follows from $\frac{1+\alpha}{2} < 1$ and the Mean Value Theorem applied to $f(x) = x^{-(1+\alpha)/2}$, i.e. for some $\theta \in (0,1)$,

$$f(x+h) - f(x) = f'(x+\theta h)h = -\left(\frac{1+\alpha}{2}\right)\frac{h}{|x+\theta h|^{(3+\alpha)/2}}, \tag{5.8}$$

with $x = \sigma^2, h = \tau^2$, and $|x+\theta h|^{-(3+\alpha)/2} \le |x|^{-(3+\alpha)/2}$. This implies

$$\begin{aligned}
|I_{\text{near},3}| &\le \|a''\|_{L^\infty}\left(\int_{-\tau}^{\tau} 2\sigma^{2-1-\alpha}\,\mathrm{d}\sigma + \int_{\tau\le|\sigma|\le\sigma_0} \tau^2\sigma^{2-3-\alpha}\,\mathrm{d}\sigma\right)\\
&= O(\tau^{2-\alpha}).
\end{aligned}$$

We focus now on $I_{\text{near},1}$.

Constant case

Here we will try to understand the following integral which appears in $I_{\text{near},1}$,

$$\begin{aligned}
J := \int_{-\sigma_0}^{\sigma_0} \frac{1}{|\sigma^2+\tau^2|^{(1+\alpha)/2}}\,\mathrm{d}\sigma &= \tau^{-\alpha}\int_{-\sigma_0/\tau}^{\sigma_0/\tau} \frac{1}{|\sigma^2+1|^{(1+\alpha)/2}}\,\mathrm{d}\sigma\\
&= 2\tau^{-\alpha}\int_{0}^{\sigma_0/\tau} \frac{1}{|\sigma^2+1|^{(1+\alpha)/2}}\,\mathrm{d}\sigma.
\end{aligned}$$

In contrast with the $\alpha = 0$ case, the integrand is in $L^1(\mathbb{R})$, so we can easily write down the following expression with an error term,

$$
\begin{aligned}
J &= 2\tau^{-\alpha}\left(\int_0^\infty \frac{\mathrm{d}\sigma}{|\sigma^2+1|^{(1+\alpha)/2}} - \int_{\sigma_0/\tau}^\infty \frac{\mathrm{d}\sigma}{|\sigma^2+1|^{(1+\alpha)/2}}\right) \\
&= 2\tau^{-\alpha}\left(\int_0^\infty \frac{\mathrm{d}\sigma}{|\sigma^2+1|^{(1+\alpha)/2}} - \int_{\sigma_0/\tau}^\infty \frac{\mathrm{d}\sigma}{\sigma^{1+\alpha}} + \int_{\sigma_0/\tau}^\infty \frac{1}{\sigma^{1+\alpha}} - \frac{1}{|\sigma^2+1|^{(1+\alpha)/2}}\,\mathrm{d}\sigma\right) \\
&= C_\alpha\tau^{-\alpha} - \frac{2}{\alpha\sigma_0^\alpha} + \mathrm{Rem},
\end{aligned}
$$

where $C_\alpha = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma}{|\sigma^2+1|^{(1+\alpha)/2}} = \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})} < \infty$ and the remainder term Rem satisfies (using (5.8) with $x = \sigma^2$, $h = 1$)

$$
|\mathrm{Rem}| \leq 2\tau^{-\alpha}\int_{\sigma_0/\tau}^\infty \frac{\mathrm{d}\sigma}{\sigma^{3+\alpha}} = 2\tau^{-\alpha}\frac{\tau^{2+\alpha}}{(2+\alpha)\sigma_0^{2+\alpha}} = O(\tau^2), \quad \tau \to 0.
$$

Hence we have that $J = C_\alpha\tau^{-\alpha} - \frac{2}{\alpha\sigma_0^\alpha} + O(\tau^2)$ as $\tau \to 0$.

Rewriting the result in original coordinates

Here we undo the transformation $s \mapsto \sigma$.

Recall that the transformation's defining equation is $g(s) = \sigma^2$. We can write $g(s) = g''(0)s^2/2 + O(s^3)$ since $s = 0$ is a global nondegenerate minimum of g with $g(0) = 0$. Hence for $\sigma > 0$ (and therefore $s > 0$), $\sigma = s\sqrt{g''(0)/2 + O(s)} = s\sqrt{g''(0)/2} + O(s^2)$ as $s \to 0^+$, by the differentiability in h of $\sqrt{g''(0)/2 + h}$. The case $\sigma < 0$ is treated similarly, leading to

$$
\sigma = \sqrt{g(s)} = \sqrt{\frac{g''(0)}{2}}s + O(s^2), \quad s \to 0,
$$

and hence $\frac{\mathrm{d}\sigma}{\mathrm{d}s} \xrightarrow[s\to 0]{} \sqrt{\frac{g''(0)}{2}}$. This constant appears often in what follows, so define the shorthand $G := \sqrt{\frac{g''(0)}{2}}$. Remembering that $a(s) = \frac{\mathrm{d}\sigma}{\mathrm{d}s}\bar{a}(\sigma)$ by definition of $\bar{a}$, this means that

$$
\bar{a}(0) = a(0)/G,
$$

which allows us to rewrite $I_{\text{near},1}$,

$$I_{\text{near},1} = \frac{a(0)}{G} C_\alpha \tau^{-\alpha} - 2\frac{a(0)}{\alpha G \sigma_0^\alpha} + O(\tau^2), \quad \tau \to 0.$$

Let us now treat $I_{\text{near},2}$. Let $0 < \sigma_1 \ll \sigma_0$, and let $s_{1-} < 0$, $s_{1+} > 0$ be the two unique numbers such that $g(s_{1\pm}) = \sigma_1^2$. Since $\bar{a}(\sigma)\,\mathrm{d}\sigma = a(s)\,\mathrm{d}s$, it is clear that $\frac{\bar{a}(\sigma)\,\mathrm{d}\sigma}{|\sigma|^{1+\alpha}} = \frac{a(s)\,\mathrm{d}s}{g(s)^{(1+\alpha)/2}}$. Hence, we only need to rewrite the other term of the difference $\frac{(\bar{a}(\sigma)-\bar{a}(0))\,\mathrm{d}\sigma}{|\sigma|^{1+\alpha}}$, which is $\bar{a}(0)\int_{\sigma_1}^{\sigma_0}\frac{\mathrm{d}\sigma}{\sigma^{1+\alpha}}$. We would like to replace the integral in σ with an integral in s. Observe that as $0 < \sigma_1 < \sigma_0$ and $0 < s_{1+} < s_+$,

$$\int_{\sigma_1}^{\sigma_0} \frac{\mathrm{d}\sigma}{\sigma^{1+\alpha}} = \frac{1}{\alpha}\left(\frac{1}{\sigma_1^\alpha} - \frac{1}{\sigma_0^\alpha}\right), \text{ and } \int_{s_{1+}}^{s_+} \frac{\mathrm{d}s}{s^{1+\alpha}} = \frac{1}{\alpha}\left(\frac{1}{s_{1+}^\alpha} - \frac{1}{s_+^\alpha}\right).$$

Thus, we have the following equality for any constant $\tilde{C}$,

$$\int_{\sigma_1}^{\sigma_0} \frac{\mathrm{d}\sigma}{\sigma^{1+\alpha}} = \tilde{C}\int_{s_{1+}}^{s_+} \frac{\mathrm{d}s}{s^{1+\alpha}} + \frac{1}{\alpha}\Bigg(\underbrace{\frac{1}{\sigma_1^\alpha} - \frac{\tilde{C}}{s_{1+}^\alpha}}_{\star} + \frac{\tilde{C}}{s_+^\alpha} - \frac{1}{\sigma_0^\alpha}\Bigg).$$

Treating s as a function $s = s(\sigma)$, the Inverse Function Theorem gives the asymptotic $s = G^{-1}\sigma + O(\sigma^2)$ for $\sigma \ll 1$. This suggests setting $\tilde{C} = G^{-\alpha}$, as then the terms marked with a star $\star$ become error terms for $\sigma_1 \ll 1$,

$$\star = \frac{1}{\sigma_1^\alpha} - \frac{1}{(Gs_{1+})^\alpha} = \frac{1}{\sigma_1^\alpha} - \frac{1}{(\sigma_1 + O(\sigma_1^2))^\alpha} \le \frac{2O(\sigma_1^2)}{\alpha\sigma_1^{1+\alpha}} = O(\sigma_1^{1-\alpha}), \quad \sigma_1 \to 0.$$

We can do a similar analysis for the integral $\int_{-\sigma_0}^{-\sigma_1}\frac{\mathrm{d}\sigma}{|\sigma|^{1+\alpha}}$, yielding

$$\int_{-\sigma_0}^{-\sigma_1} \frac{\mathrm{d}\sigma}{|\sigma|^{1+\alpha}} = \frac{1}{G^\alpha}\int_{s_-}^{s_{1-}} \frac{\mathrm{d}s}{|s|^{1+\alpha}} + \frac{1}{\alpha}\Bigg(\underbrace{\frac{1}{\sigma_1^\alpha} - \frac{1}{|Gs_{1-}|^\alpha}}_{=O(\sigma_1^{1-\alpha})} + \frac{1}{|Gs_-|^\alpha} - \frac{1}{\sigma_0^\alpha}\Bigg),$$

which together yield (as $\int_{-\sigma_0}^{-\sigma_1}\frac{\mathrm{d}\sigma}{|\sigma|^{1+\alpha}} = \int_{\sigma_1}^{\sigma_0}\frac{\mathrm{d}\sigma}{|\sigma|^{1+\alpha}}$)

$$\begin{aligned} &2\int_{\sigma_0}^{\sigma_1} \frac{\mathrm{d}\sigma}{|\sigma|^{1+\alpha}} \\ &= \int_{s_{1+}}^{s_+} \frac{\mathrm{d}s}{G^\alpha|s|^{1+\alpha}} + \int_{s_-}^{s_{1-}} \frac{\mathrm{d}s}{G^\alpha|s|^{1+\alpha}} + \frac{1}{\alpha}\left(\frac{1}{|Gs_+|^\alpha} + \frac{1}{|Gs_-|^\alpha} - \frac{2}{|\sigma_0|^\alpha}\right) + O(\sigma_1^{1-\alpha}). \end{aligned}$$

Hence, we rewrite $I_{\text{near},2}$ as follows,

$$\begin{aligned}
I_{\text{near},2} &= \int_{-\sigma_0}^{\sigma_0} \frac{\bar{a}(\sigma)-\bar{a}(0)}{|\sigma|^{1+\alpha}}\,\mathrm{d}\sigma \\
&= \lim_{\sigma_1\to 0}\left(\int_{\sigma_1}^{\sigma_0}\frac{\bar{a}(\sigma)}{|\sigma|^{1+\alpha}}\,\mathrm{d}\sigma + \int_{-\sigma_0}^{-\sigma_1}\frac{\bar{a}(\sigma)}{|\sigma|^{1+\alpha}}\,\mathrm{d}\sigma - 2\int_{\sigma_1}^{\sigma_0}\frac{\bar{a}(0)}{|\sigma|^{1+\alpha}}\,\mathrm{d}\sigma\right) \\
&= \lim_{\sigma_1\to 0}\left(\int_{s_{1+}}^{s_+}\frac{a(s)}{|g(s)|^{1+\alpha}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s\right. \\
&\qquad \left. + \int_{s_-}^{s_{1-}}\frac{a(s)}{|g(s)|^{1+\alpha}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s + O(\sigma_1^{1-\alpha})\right) \\
&\qquad - \frac{a(0)}{\alpha G^{1+\alpha}s_+^{\alpha}} - \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^{\alpha}} + 2\frac{a(0)}{\alpha G\sigma_0^{\alpha}} \\
&= \int_{s_-}^{s_+}\frac{a(s)}{|g(s)|^{1+\alpha}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s \\
&\qquad - \frac{a(0)}{\alpha G^{1+\alpha}s_+^{\alpha}} - \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^{\alpha}} + 2\frac{a(0)}{\alpha G\sigma_0^{\alpha}} + O(\sigma_1^{1-\alpha}).
\end{aligned} \tag{5.9}$$

As luck would have it, the term $2\frac{a(0)}{\alpha G\sigma_0^{\alpha}}$ here in (5.9) exactly cancels with the term with $-2\frac{a(0)}{\alpha G\sigma_0^{\alpha}}$ in the equation (5.10) for $I_{\text{near},1}$. We therefore can write I_{near} as follows,

$$\begin{aligned}
I_{\text{near}} = \frac{a(0)}{G}C_\alpha\tau^{-\alpha} &+ \int_{s_-}^{s_+}\frac{a(s)}{|g(s)|^{1+\alpha}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s \\
&- \frac{a(0)}{\alpha G^{1+\alpha}s_+^{\alpha}} - \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^{\alpha}} + O(\tau^{2-\alpha}), \quad \tau\to 0.
\end{aligned} \tag{5.10}$$

The full result

To finish, we need to also consider I_{far}. Recall from (5.7) that $A_{\text{far}} = \mathbb{T}\setminus(s_-, s_+)$. Note that with $s_\pm$ fixed, $g(s)^{-(1+\alpha)/2}$ is $L_s^\infty(A_{\text{far}})$, and the following error estimate holds, since $\frac{a(s)}{|g(s)+\tau^2|^{(1+\alpha)/2}}$ is smooth in $\tau \ll 1$:

$$\int_{A_{\text{far}}}\frac{a(s)\,\mathrm{d}s}{|g(s)+\tau^2|^{(1+\alpha)/2}} = \int_{A_{\text{far}}}\frac{a(s)\,\mathrm{d}s}{|g(s)|^{(1+\alpha)/2}} + O(\tau^2).$$

We also have the following easy computation,

$$\int_{A_{\text{far}}}\frac{\mathrm{d}s}{|s|^{1+\alpha}} = \left(\int_{-1/2}^{s_-} + \int_{s_+}^{1/2}\right)\frac{\mathrm{d}s}{|s|^{1+\alpha}} = \frac{-2^{1+\alpha}}{\alpha} + \frac{1}{\alpha s_+^{\alpha}} + \frac{1}{\alpha|s_-|^{\alpha}}.$$

This yields

$$
\begin{aligned}
I_{\mathrm{far}} &= \int_{A_{\mathrm{far}}} \frac{a(s)}{|g(s)+\tau^2|^{(1+\alpha)/2}}\,\mathrm{d}s \\
&= \int_{A_{\mathrm{far}}} \frac{a(s)}{|g(s)|^{(1+\alpha)/2}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s \\
&\quad + \frac{a(0)}{G^{1+\alpha}} \int_{A_{\mathrm{far}}} \frac{1}{|s|^{1+\alpha}}\,\mathrm{d}s + O(\tau^2) \\
&= \int_{A_{\mathrm{far}}} \frac{a(s)}{|g(s)|^{(1+\alpha)/2}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s \\
&\quad \frac{-2^{1+\alpha}a(0)}{\alpha G^{1+\alpha}} + \frac{a(0)}{\alpha G^{1+\alpha} s_+^\alpha} + \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^\alpha} + O(\tau^2).
\end{aligned} \tag{5.11}
$$

The terms $\frac{a(0)}{\alpha G^{1+\alpha} s_+^\alpha} + \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^\alpha}$ in (5.11) cancel the terms $-\frac{a(0)}{\alpha G^{1+\alpha} s_+^\alpha} - \frac{a(0)}{\alpha G^{1+\alpha}|s_-|^\alpha}$ in (5.10), leaving an expression that does not depend on $s_\pm$. Therefore, we finally arrive at a complete asymptotic for I,

$$
\begin{aligned}
I &= \frac{a(0)}{G} C_\alpha \tau^{-\alpha} - \frac{2^{1+\alpha}a(0)}{\alpha G^{1+\alpha}} \\
&\quad + \int_{-1/2}^{1/2} \frac{a(s)}{|g(s)|^{(1+\alpha)/2}} - \frac{a(0)}{G^{1+\alpha}|s|^{1+\alpha}}\,\mathrm{d}s + O(\tau^{2-\alpha}), \quad \tau \to 0.
\end{aligned}
$$

Since a, g are period 1 functions, defining

$$
b_\alpha := \int_{-1/2}^{1/2} \frac{1}{|s|^{1+\alpha}} - \frac{\pi^{1+\alpha}}{|\sin(\pi s)|^{1+\alpha}}\,\mathrm{d}s \in \mathbb{R},
$$

we can write

$$
\begin{aligned}
I &= \frac{a(0)}{G} C_\alpha \tau^{-\alpha} - \frac{a(0)(\alpha^{-1}2^{1+\alpha} + b_\alpha)}{G^{1+\alpha}} \\
&\quad + \int_{s\in\mathbb{T}} \frac{a(s)}{|g(s)|^{(1+\alpha)/2}} - \frac{\pi^{1+\alpha}a(0)}{G^{1+\alpha}|\sin(\pi s)|^{1+\alpha}}\,\mathrm{d}s + O(\tau^{2-\alpha}), \quad \tau \to 0,
\end{aligned}
$$

which is the claimed result. □

We also write down the following corollary.

Corollary 5.9. *Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $\delta \in (-\delta_0, \delta_0)$ sufficiently small, let $H = H(\delta)$ denote the following family of integrals,*

$$
H = \int_{s\in\mathbb{T}} \left(\frac{a(s)}{|g(s,\delta)|^{(1+\alpha)/2}} - \frac{a(0)}{G(\delta)^{1+\alpha}|\operatorname{Sin} s|^{1+\alpha}} \right) \mathrm{d}s,
$$

where $a = a(s) \in C^\infty(\mathbb{T})$, $g = g(s,\delta) \in C^\infty(\mathbb{T} \times [0,\infty))$, and g has a unique global minimum that is non-degenerate with $\partial_s^2 g(0,\cdot) > c > 0$ for a constant c independent of δ, and

$$\operatorname{argmin} g(\cdot,\delta) = 0, \quad g(0,\delta) = \min g(\cdot,\delta) = 0$$

and $G(\delta) := \sqrt{\partial_s^2 g(0,\delta)/2}$. Then we have the first order Taylor expansion $H(\tau) = H(0) + H'(0)\delta + O(\delta^2)$ for $\tau \ll 1$, with

$$H'(0) = \int_{s\in\mathbb{T}} \left(\frac{a(s)\partial_\delta g(s,0)(-1-\alpha)}{|g(s,0)|^{(3+\alpha)/2}} - \frac{\partial_\delta (G^{-1-\alpha})(0)a(0)}{|\operatorname{Sin} s|^{1+\alpha}} \right) \mathrm{d}s.$$

Remark 5.10 (logarithmic asymptotic for $\alpha = 1$)*.* The above does not cover the case $\alpha = 1$. This has been computed in [28], and we paraphrase it here for completeness.

For $\tau > 0$, let $I = I(\tau)$ denote the following family of integrals,

$$I = \int_{s\in\mathbb{T}} \frac{a(s)}{|g(s) + \tau^2|^{1/2}} \, \mathrm{d}s,$$

where $a = a(s), g = g(s) \in C^\infty(\mathbb{T})$ and g has a unique global minimum that is non-degenerate with

$$\operatorname{argmin} g = 0, \quad g(0) = \min g = 0.$$

Then we have the asymptotic expansion as $\tau \to 0$,

$$\begin{aligned} I = {} & \frac{a(0)}{2G} \log \tau + \frac{a(0)(\log \pi + \log G + b)}{G} \\ & + \int_{s\in\mathbb{T}} \frac{a(s)}{|g(s)|^{1/2}} - \frac{a(0)}{G|\operatorname{Sin} s|} \, \mathrm{d}s + O(\tau^2 \log \tau), \quad \tau \to 0, \end{aligned}$$

where:

1. $\operatorname{Sin} s := \sin(\pi s)/\pi$,
2. b is the constant $b := \int_{-1/2}^{1/2} \frac{1}{|s|} - \frac{1}{|\operatorname{Sin} s|} \, \mathrm{d}s < \infty$,
3. G is the constant $G := \sqrt{g''(0)/2}$ which is well-defined since $g''(0) > 0$, and
4. the $O(\tau^2 \log \tau)$ constant depends on $W^{3,\infty}$ norms of a and g.

Chapter 6

Curves in the transition region of Almost-Sharp Fronts

In this chapter, we demonstrate that ASF solutions to our generalised SQG equation shares some similarities with ASF solutions to the SQG equation. Firstly, we show that a generic curve transported by an almost-sharp front evolves like a sharp front up to $O(\delta^{1-\alpha})$ errors. We also provide an elementary proof of a slightly weaker result. Finally, we show that the analogue of a spine curve from [26] can also be defined in our setting, and its evolution has the better behaved $O(\delta^{2-\alpha})$ error.

6.1 Evolution of compatible curves

We have seen that a sharp front solution to (1.1) is completely determined by the evolution of a curve. For an almost-sharp front, any open region where θ is constant will remain such a region for short times, so the evolution is fully specified by the evolution of the transition region of size $O(\delta)$. A first step in understanding their evolution comes from understanding how compatible curves are transported by the equation (1.1) in the regime $\delta \ll 1$, which we now address.

We will prove the main result of this section, Theorem 6.2 by relying on a fractional Leibniz rule (6.2). Then, we will give an elementary lemma (Lemma 6.3) that replaces the more complicated (6.2) at the cost of a small loss in the error term. The proof method of the lemma is similar to a simpler lemma (which can be found in [25] for instance) which we shall prove first, since we also need it for the proof of the full result. The author could not find Lemma 6.3 in the literature.

Lemma 6.1. *Let $0 < s < 1/2$ and suppose $A \subseteq \mathbb{R}^d$ is a bounded set with C^2*

boundary in $\mathbb{R}^d$*. Then* $\mathbf{1}_A \in H^s(\mathbb{R}^d)$ *with*

$$\|\Lambda^s \mathbf{1}_A\|_{L^2}^2 \lesssim_{d,s} |A|^{1-2s}.$$

Proof. We bound the Gagliardo seminorm $[\mathbf{1}_A]_{H^s}$ directly, which is known (see for instance [54]) to be equal to $\|\Lambda^s \mathbf{1}_A\|_{L^2}$ up to a constant depending on s only. By definition,

$$[\mathbf{1}_A]_{H^s}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathbf{1}_A(x) - \mathbf{1}_A(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x = 2 \int_{x \in A} \int_{y \in A^c} \frac{\mathrm{d}y \, \mathrm{d}x}{|x-y|^{d+2s}}.$$

Writing out a 'layer cake' decomposition(see for example [46, page 26]) with μ the $2d$-dimensional Lebesgue measure, we obtain

$$\begin{aligned}
[\mathbf{1}_A]_{H^s}^2 &= \int_0^\infty \mu \left[x \in A, y \in A^c : |x-y| < \frac{1}{t^{1/(d+2s)}} \right] \mathrm{d}t \\
&=: \int_0^\infty m\left(\frac{1}{t^{1/(d+2s)}} \right) \mathrm{d}t,
\end{aligned}$$

where $m(t)$ is the function

$$m(t) := \mu[x \in A, y \in A^c : |x-y| < t].$$

For any set $U \subseteq \mathbb{R}^d$, define $(U)_\varepsilon = \{x : d(x, U) < \varepsilon\}$.

Performing the change of variables $\tau = \frac{1}{t^{1/(2+2s)}}$, $\mathrm{d}t = (d+2s)\tau^{-d-2s-1} \, \mathrm{d}\tau$, we can write

$$\begin{aligned}
[\mathbf{1}_A]_{H^s}^2 &= (2d+4s) \int_0^\infty \frac{m(\tau)}{\tau^{d+2s+1}} \, \mathrm{d}\tau \\
&\lesssim_{d,s} \int_0^{\varepsilon_0} \frac{m(\tau)}{\tau^{d+2s+1}} \, \mathrm{d}\tau + \int_{\varepsilon_0}^\infty \frac{m(\tau)}{\tau^{d+2s+1}} \, \mathrm{d}\tau \\
&= I_1 + I_2,
\end{aligned}$$

for some ϵ_0 to be chosen as follows. We can bound $m(\tau)$ using the inclusion

$$\{x \in A, y \in A^c : |x-y| < \tau\} \subset \{x \in A, y \in B_{\mathbb{R}^d}(x, \tau) : d(x, \partial A) < \tau\},$$

where $B_{\mathbb{R}^d}(x,\tau)$ is the ball around x of radius τ, which implies

$$\begin{aligned} m(\tau) &= \mu[x \in A, y \in A^c : |x-y| < \tau] \\ &\leq \mu[x \in A, y \in B_{\mathbb{R}^d}(x,\tau) : d(x,\partial A) < \tau,] \\ &\lesssim |B_{\mathbb{R}^d}(0,1)|\tau^{d+1}, \end{aligned}$$

since $(\partial A)_\tau$ is $O(\tau)$ by the C^2 regularity of the boundary. But for $\tau > |A|$, the following easier bound is better,

$$\begin{aligned} m(\tau) &= \mu[x \in A, y \in A^c : |x-y| < \tau] \\ &\leq \mu\left[x \in A : y \in B_{\mathbb{R}^d}(x,\tau)\right] \\ &\lesssim |B_{\mathbb{R}^d}(0,1)||A|\tau^d. \end{aligned}$$

For the optimal bound, we should choose $\epsilon_0 = |A|$ to define I_1, I_2, giving

$$I_1 \lesssim_d \int_0^{|A|} \frac{\mathrm{d}\tau}{\tau^{2s}} = \frac{1}{1-2s}|A|^{1-2s} \lesssim_s |A|^{1-2s},$$

and

$$I_2 \lesssim_d |A| \int_{|A|}^{\infty} \frac{\mathrm{d}\tau}{\tau^{2s+1}} = |A|\frac{|A|^{-2s}}{2s} \lesssim_s |A|^{1-2s}.$$

Combining the bounds for I_1 and I_2 gives the result. □

Armed with this inequality, we can now present the main result of this section.

Theorem 6.2. *Suppose that θ is an ASF solution to (1.1), and z is a compatible curve as defined in (5.2). Then as z is transported by u, it evolves (in the weak sense) by the sharp front equation up to $O(\delta^{1-\alpha})$ errors,*

$$\partial_t z \cdot N = \left(-\int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z)\, \mathrm{d}s_* \right) \cdot N + O(\delta^{1-\alpha}).$$

Proof. The strategy of this proof is the same as [19]. For brevity of notation, we shall in this proof write

$$(\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{mid}}, \mathbf{1}_{\text{out}}) := (\mathbf{1}_{A^z_{\text{in}}}, \mathbf{1}_{A^z_{\text{mid}}}, \mathbf{1}_{A^z_{\text{out}}}).$$

As in the proof of Proposition 2.4, the term $\iint_{\mathbb{R}^2\times[0,\infty)} \partial_t\phi\theta$ brings out the time derivative of the C^2 boundary curve(which has bounded curvature) as follows: since the set A_{mid} has measure $O(\delta)$, we see that if $z^\delta = z + \delta N$ parameterises the boundary of A_{in},

$$\iint_{\mathbb{R}^2\times[0,\infty]} \partial_t\phi\theta \,\mathrm{d}x\,\mathrm{d}t = \iint_{\{x\in\mathbb{R}^2, t\geq 0: x\in A_{\mathrm{in}}(t)\}} \partial_t\phi \,\mathrm{d}x\,\mathrm{d}t + O(\delta)$$
$$= \int_0^\infty \int_{\mathbb{T}} \partial_t(z^\delta)\nu^3\phi(z^\delta, t)\,\mathrm{d}s\,\mathrm{d}t + O(\delta).$$

Here, ν^3 is the outward normal ν's third component as a vector in (x^1, x^2, t)-space, and

$$\partial_t(z^\delta)\nu^3 = \partial_t(z^\delta)\cdot\partial_s((z^\delta)^\perp) = (L - \delta L\kappa)\partial_t(z+\delta N)\cdot\partial_s z^\perp$$
$$= L\partial_t z\cdot N + O(\delta),$$

where we used our definition of a compatible curve. Hence, as $\phi \in C^1$ and $\mathrm{d}l = L\,\mathrm{d}s$ for the uniform speed parameterised curve z, writing ∂A for the curve parameterised by z, we have

$$\iint_{\mathbb{R}^2\times[0,\infty]} \partial_t\phi\theta\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\infty\int_{\partial A} \phi(z)\partial_t z\cdot N\,\mathrm{d}l\,\mathrm{d}t + O(\delta).$$

We now treat the second term $\iint_{\mathbb{R}^2\times[0,\infty]} u\cdot\nabla\phi\theta:$ observe the following decomposition, where we have written $u = \nabla^\perp K * \theta = (\nabla^\perp K) * \theta$ as a convolution of θ with the kernel $\nabla^\perp K$, and used the bilinearity of $(f,g)\mapsto \int_{\mathbb{R}^2}\nabla^\perp K * f\cdot\nabla\phi g$, and $\theta = \theta\cdot(\mathbf{1}_{\mathrm{in}} + \mathbf{1}_{\mathrm{mid}} + \mathbf{1}_{\mathrm{out}}) = \theta\mathbf{1}_{\mathrm{mid}} + \mathbf{1}_{\mathrm{in}}$:

$$\begin{aligned}
&\int_{\mathbb{R}^2} u\cdot\nabla\phi\theta \\
&= \int_{\mathbb{R}^2}\nabla^\perp K * \mathbf{1}_{\mathrm{in}}\cdot\nabla\phi\mathbf{1}_{\mathrm{in}} + \int_{\mathbb{R}^2}\nabla^\perp K * \mathbf{1}_{\mathrm{in}}\cdot\nabla\phi\theta\mathbf{1}_{\mathrm{mid}} \\
&\quad + \int_{\mathbb{R}^2}\nabla^\perp K * (\theta\mathbf{1}_{\mathrm{mid}})\cdot\nabla\phi\theta\mathbf{1}_{\mathrm{mid}} + \int_{\mathbb{R}^2}\nabla^\perp K * (\theta\mathbf{1}_{\mathrm{mid}})\cdot\nabla\phi\mathbf{1}_{\mathrm{in}} \\
&=: (\mathrm{EVO}) + (\mathrm{A}) + (\mathrm{B}) + (\mathrm{C}) \\
&= (\mathrm{EVO}) + [(\mathrm{A}) + (\mathrm{B})] + [(\mathrm{B}) + (\mathrm{C})] - (\mathrm{B}).
\end{aligned} \tag{6.1}$$

We will estimate separately each of the 4 terms in the last line of (6.1). Up to $O(\delta^{1-\alpha})$ errors, (EVO) will give us the evolution term, and the square-bracketed terms will use the C^ε regularity of $\theta = \theta\mathbf{1}_{\mathrm{mid}} + \mathbf{1}_{\mathrm{in}}$ that is not available when estimating (A) or (C) alone.

1. Control on $[(\mathrm{A})+(\mathrm{B})]$. We proceed by splitting the kernel K,

$$\begin{aligned}
[(\mathrm{A})+(\mathrm{B})] &= \int_{\mathbb{R}^2} \nabla^\perp K * \theta \cdot \nabla\phi \mathbf{1}_{\mathrm{mid}} \\
&= \int_{\mathbb{R}^2} ((\nabla^\perp K)\mathbf{1}_{|\cdot|>\delta}) * \theta \cdot \nabla\phi \mathbf{1}_{\mathrm{mid}} \\
&\quad + \int_{\mathbb{R}^2} ((\nabla^\perp K)\mathbf{1}_{|\cdot|<\delta}) * \theta \cdot \nabla\phi \mathbf{1}_{\mathrm{mid}} \\
&=: I_1 + I_2.
\end{aligned}$$

We note the bounds

$$\begin{aligned}
|((\nabla^\perp K)\mathbf{1}_{|\cdot|>\delta}) * \theta(x)| &\lesssim \|\theta\|_{L^\infty} \int_{r=\delta}^\infty \frac{r\,\mathrm{d}r}{r^{2+\alpha}} \lesssim_{\theta,\alpha} \delta^{-\alpha}, \text{ and} \\
|((\nabla^\perp K)\mathbf{1}_{|\cdot|<\delta}) * \theta(x)| &= \left| \int_{|y|<\delta} K(y)\theta(x-y)\,\mathrm{d}y \right| \\
&= \left| \int_{|y|<\delta} K(y)|y|^{\alpha'} \left(\frac{\theta(x-y)-\theta(x)}{|y|^{\alpha'}} \right) \mathrm{d}y \right| \\
&\lesssim [\theta]_{C^{\alpha'}} \int_{r=0}^{\delta} \frac{r\,\mathrm{d}r}{r^{2+\alpha-\alpha'}} \\
&\lesssim [\theta]_{C^{\alpha'}} \delta^{\alpha'-\alpha} \\
&= O(\delta^{-\alpha}),
\end{aligned}$$

where in the second inequality, we used the regularity assumption $[\theta]_{C^{\alpha'}} \lesssim \delta^{-\alpha'}$ for some $\alpha' > \alpha$. Hence both I_1 and I_2 are integrals of $O(\delta^{-\alpha})$ functions that have support of size $O(\delta)$, due to the $\mathbf{1}_{\mathrm{mid}}$ term. Therefore, $(\mathrm{A})+(\mathrm{B}) = O(\delta^{1-\alpha})$.

2. Control on $[(\mathrm{B})+(\mathrm{C})]$. Here, notice that

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla^\perp K * f \cdot g &= -\int_{\mathbb{R}^2} \partial_{x^2} K * f g^1 + \int_{\mathbb{R}^2} \partial_{x^1} K * f g^2 \\
&= -\int_{\mathbb{R}^2} f \partial_{x^2} K * g^1 + \int_{\mathbb{R}^2} f \partial_{x^1} K * g^2.
\end{aligned}$$

The important feature is that the two kernels $\partial_{x^1} K(-x), \partial_{x^2} K(-x)$ have the same $-2-\alpha$ homogeneity as $\nabla^\perp K$, and have mean zero on the unit sphere. Hence, with $f = \theta \mathbf{1}_{\mathrm{mid}}$ and $g = \nabla\phi\theta \in C^\varepsilon$, we can repeat the proof as for $[(\mathrm{A})+(\mathrm{B})]$, obtaining the same $O(\delta^{1-\alpha})$ estimate.

3. Control on (B). Writing $R = \nabla\Lambda^{-1}$ for the vector of Riesz transforms (see for

instance [76]), we have $\nabla^\perp K * f = R^\perp \Lambda^\alpha f$, so that

$$
\begin{aligned}
|(\mathrm{B})| &= \left| \int_{\mathbb{R}^2} \nabla^\perp K * (\theta \mathbf{1}_{\mathrm{mid}}) \cdot (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right| \\
&= \left| \int_{\mathbb{R}^2} R^\perp \Lambda^\alpha (\theta \mathbf{1}_{\mathrm{mid}}) \cdot (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right| \\
&= \left| \int_{\mathbb{R}^2} R^\perp \Lambda^{\alpha/2} (\theta \mathbf{1}_{\mathrm{mid}}) \cdot \Lambda^{\alpha/2} (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right| \\
&\lesssim \left\| R^\perp \Lambda^{\alpha/2} (\theta \mathbf{1}_{\mathrm{mid}}) \right\|_{L^2} \left\| \Lambda^{\alpha/2} (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right\|_{L^2} \\
&\lesssim \left\| \Lambda^{\alpha/2} (\theta \mathbf{1}_{\mathrm{mid}}) \right\|_{L^2} \left\| \Lambda^{\alpha/2} (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right\|_{L^2},
\end{aligned}
$$

where the last line is by the boundedness of $R^\perp : L^2 \to L^2$. To bound these terms, we will use the following fractional Leibniz rule of e.g. [41],

$$
\|\Lambda^s (fg)\|_{L^2} \lesssim \|\Lambda^s f\|_{L^\infty} \|g\|_{L^2} + \|f\|_{L^\infty} \|\Lambda^s g\|_{L^2}, \tag{6.2}
$$

and the following easy estimate that comes from bounding the following two terms separately (similarly to the earlier part of this proof) $\Lambda^s f(x) = \int_{\mathbb{R}^2} \frac{f(x)-f(y)}{|x-y|^{2+s}} \, \mathrm{d}y = \left(\int_{|x-y| \le \delta} + \int_{|x-y| > \delta} \right) \frac{f(x)-f(y)}{|x-y|^{2+s}} \, \mathrm{d}y$,

$$
\|\Lambda^s f(x)\|_{L^\infty} \lesssim \delta^\epsilon [f]_{C^{s+\epsilon}} + \delta^{-s} \|f\|_{L^\infty}.
$$

By interpolation and the assumption $|\nabla \theta| \lesssim \frac{1}{\delta}$ in Definition 5.1, $[f]_{C^{s+\epsilon}} \lesssim \delta^{-s-\epsilon}$ for $s + \epsilon \le 1$. Setting $s = \alpha/2$, $g = \mathbf{1}_{\mathrm{mid}}$ (note $\|g\|_{L^2} = \delta^{1/2}$) and $f = \theta$ or $\nabla \phi \theta$, we obtain

$$
\|\Lambda^{\alpha/2} (\theta \mathbf{1}_{\mathrm{mid}})\|_{L^2} \lesssim \delta^{(1-\alpha)/2},
$$

and also

$$
\left\| \Lambda^{\alpha/2} (\nabla \phi \theta \mathbf{1}_{\mathrm{mid}}) \right\|_{L^2} \lesssim \delta^{(1-\alpha)/2}.
$$

Together, these inequalities prove that $|(\mathrm{B})| \lesssim \delta^{1-\alpha}$.

4. Evolution term in (EVO). By following the proof of the analogous sharp front result (2.4) and using $z^\delta = z + \delta N$ again,

$$
\begin{aligned}
I_{\mathrm{in}} &= \int_{\mathbb{T}} \phi(z^\delta) \left(\int_{\mathbb{T}} K(z^\delta - z_*^\delta) \partial_s z_*^\delta \cdot \partial_s (z_*^\delta)^\perp \, \mathrm{d}s_* \right) \mathrm{d}s \\
&= \int_{s \in I} \phi(z) \left(\int_{\mathbb{T}} K(z - z_*) (\partial_s z_* - \partial_s z) \, \mathrm{d}s_* \right) \cdot N \, \mathrm{d}s + O(\delta).
\end{aligned}
$$

This last line follows from a simple application of the Mean Value Theorem, treating δN as an increment. This completes the computation of the required inequalities, and the result follows. $\square$

In the above proof, we relied on a fractional Leibniz rule (6.2). The following lemma can serve as a weak replacement:

Lemma 6.3 (Hölder-Indicator Leibniz Rule)**.** *Let $0 < s < 1/2$, $s < s'$, suppose $A \subseteq \mathbb{R}^d$ is a bounded set with C^2 boundary in $\mathbb{R}^d$, Let $f \in C^{s'}(A)$ be an s'-Hölder function. Then the extension of f by zero, $f\mathbf{1}_A$ belongs to $H^s(\mathbb{R}^d)$ with*

$$\|\Lambda^s(f\mathbf{1}_A)\|^2_{L^2(\mathbb{R}^d)} \lesssim_{d,s} \|f\|^2_{L^\infty(A)}|A|^{1-2s} + [f]^2_{C^{s'}(A)}|A|^{1-\frac{2(s-s')}{d}}.$$

Proof. We again bound the Gagliardo seminorm $[f\mathbf{1}_A]^2_{H^s}$, similarly to the proof of Lemma 6.1. We have

$$\begin{aligned}
[f\mathbf{1}_A]^2_{H^s} &= 2\int_{A^c}\int_A \frac{|(f\mathbf{1}_A)(x) - (f\mathbf{1}_A)(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_A\int_A \frac{|(f\mathbf{1}_A)(x) - (f\mathbf{1}_A)(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_{A^c}\int_A \frac{|f(x)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_A\int_A \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Using the inclusion,

$$\left\{x \in A, y \in A^c \colon \frac{|f(x)|^2}{|x-y|^{d+2s}} > t\right\} \subseteq \left\{x \in A, y \in A^c \colon \frac{\|f\|^2_{L^\infty}}{|x-y|^{d+2s}} > t\right\},$$

one can easily follow the proof of Lemma 6.1 for when only one of x, y is in A to obtain

$$\int_{A^c}\int_A \frac{|f(x)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \lesssim_{d,s} \|f\|^2_{L^\infty}|A|^{1-2s},$$

but unlike Lemma 6.1, we do have contributions from when $(x,y) \in A \times A$. That is, we need to obtain a bound on the following integral,

$$I_A := \int_A\int_A \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Using the layer cake decomposition again with

$$\tilde{m}(t) := \mu(x, y \in A, y \in A : |x-y| < t),$$

we can estimate I_A as follows:

$$
\begin{aligned}
I_A &= \int_0^\infty \mu\left[x \in A, y \in A : \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} > t\right] \mathrm{d}t \\
&\leq \int_0^\infty \mu\left[x \in A, y \in A : \frac{[f]_{C^{s'}}^2}{|x-y|^{d+2(s-s')}} > t\right] \mathrm{d}t \\
&= \int_0^\infty \mu\left[x \in A, y \in A : |x-y| < \left([f]_{C^\varepsilon}^2/t\right)^{\frac{1}{d+2(s-s')}}\right] \mathrm{d}t \\
&= \int_0^\infty \tilde{m}\left(([f]_{C^{s'}}^2/t)^{\frac{1}{d+2(s-s')}}\right) \mathrm{d}t \\
&\lesssim_{d,s,s'} [f]_{C^{s'}}^2 \int_0^\infty \frac{\tilde{m}(\tau)}{\tau^{d+2(s-s')+1}} \,\mathrm{d}\tau,
\end{aligned}
$$

where in the last line we have changed variables $\tau = ([f]_{C^{s'}}^2/t)^{\frac{1}{d+2(s-s')}}$ and ignored some constants. Observe now that by reasoning similarly to the proof of Lemma 6.1,

$$\tilde{m}(\tau) \lesssim_d \min(|A|^2, |A|\tau^d).$$

Hence, the optimal bound is obtained by splitting the integration region $\tau > 0$ into the sets $\tau \in [0, |A|^{1/d}]$ and $\tau \in [|A|^{1/d}, \infty]$, which yields the following inequalities,

$$
\begin{aligned}
I_A &\lesssim [f]_{C^{s'}}^2 |A| \int_0^{|A|^{1/d}} \frac{\mathrm{d}\tau}{\tau^{2(s-s')+1}} + [f]_{C^{s'}}^2 |A|^2 \int_{|A|^{1/d}}^\infty \frac{\mathrm{d}\tau}{\tau^{d+2(s-s')+1}} \\
&= [f]_{C^{s'}}^2 \left(|A|^{1-\frac{2(s-s')}{d}} + |A|^{2-\frac{d+2(s-s')}{d}}\right) \\
&\lesssim [f]_{C^{s'}}^2 |A|^{1-\frac{2(s-s')}{d}},
\end{aligned}
$$

so long as $s' > s$. As there is no contribution to $[f\mathbf{1}_A]_{H^s}^2$ when $(x,y) \in A^c \times A^c$, this concludes the proof. $\square$

In view of Lemma 6.1, Lemma 6.3 is reminiscent of the fractional Leibniz rule (6.2). Lemma 6.3 instead gives us the slightly weaker result for any $\theta \in C^{s+\epsilon}(\mathbb{R}^2)$,

$$\|\Lambda^s(\theta \mathbf{1}_A)\|_{L^2} \lesssim \delta^{\frac{1}{2}-s-\epsilon/2},$$

or $\|\Lambda^s(\theta\mathbf{1}_A)\|_{L^2} \leq \delta^{\frac{1}{2}-\frac{2s}{d}-\epsilon(1-\frac{1}{d})}$ for arbitrary dimensions d. This produces an elementary proof of the following slightly weaker result:

Proposition 6.4. *Suppose that θ is an ASF solution to (1.1), and z is a compatible curve as defined in (5.2). Then for any $\alpha' > \alpha$, as z is transported by u, it evolves* *(in the weak sense) by the sharp front equation up to $O(\delta^{1-\alpha'})$ errors,*

$$\partial_t z \cdot N = \left(-\int_{\mathbb{T}} K(z - z_*)(\partial_s z_* - \partial_s z) \,\mathrm{d}s_* \right) \cdot N + O(\delta^{1-\alpha'}).$$

6.2 The spine of an almost-sharp front

Here we introduce the concept of the spine, first considered by Fefferman, Luli and Rodrigo in [26] to understand almost-sharp fronts of SQG. Our construction closely follows [26], but adapted for boundaries that are not necessarily graphs. Instead of having pre-determined Cartesian coordinates, we will have to use the fact that our definition of an almost-sharp front comes equipped with at least one compatible curve.

To simplify the following calculations, assume without loss of generality Ω is given by (5.6).

Definition 6.5. Suppose an almost-sharp front has tubular neighbourhood coordinates (see Section 5.1.1) (s, ξ) for the transition region, induced by the compatible curve z.

We say that the curve S is a spine for the almost-sharp front with base curve z if S is also a compatible curve, and there is a C^2 function of the uniform speed parameter $f = f(s)$ taking values in $[-C^z, C^z]$ such that

$$\int_{-C^z}^{C^z} (\xi_* - f(s_*))\partial_\xi \Omega_* \,\mathrm{d}\xi_* = 0,$$

or equivalently by the choice $\Omega|_{\xi=\pm C^z} = \pm 1/2$, $f(s_*) = -\int_{-C^z}^{C^z} \Omega_* \,\mathrm{d}\xi_*$, and the corresponding spine is the curve S given in (s, ξ) coordinates as $\xi = f(s)$, that is:

$$S(s) = z(s) + \delta f(s) N(s).$$

The function f acts as a correction so that, for example, the base curve is also a spine if $f = 0$.

An immediate consequence of Definition 6.5 by integrating by parts is the following cancellation property for any constant $C \geq \|f\|_{L^\infty}$,

$$0 = \int_{\xi_*=f-C-C^z}^{f+C+C^z} \Omega_* \,\mathrm{d}\xi_*, \tag{6.3}$$

where Ω is continuously extended to be constant on $|\xi| \geq C^z$ past the geometrically

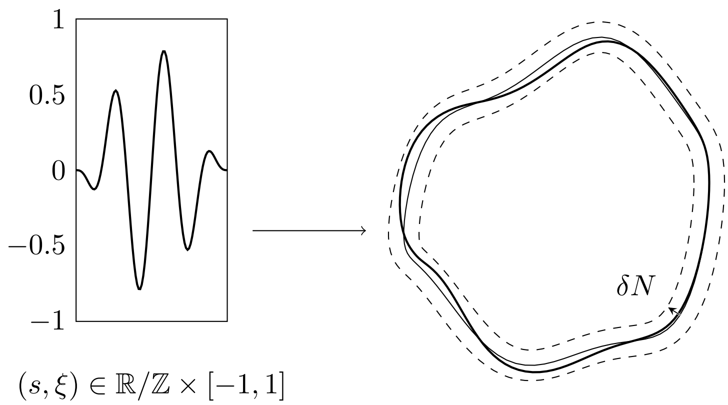

Figure 6.1: An illustration (thick curve) of a function $\xi = f(s)$ mapped to the tubular neighbourhood of a curve (thin curve). If it satisfies the equation $0 = \int_{C^z}^{C^z}(\xi_* - f(s_*))\partial_\xi \Omega_* \,\mathrm{d}\xi_*$, then it defines the spine curve from Definition 6.5.

significant range that defines the tubular neighbourhood. Indeed, for $D \gg 1$,

$$\int_{f-D}^{f+D} \Omega_* \,\mathrm{d}\xi_* = \underbrace{\xi_*\Omega_*\Big|_{f-D}^{f+D}}_{=-f} - \underbrace{\int_{f-D}^{f+D} \xi_*\partial_\xi\Omega_* \,\mathrm{d}\xi_*}_{=\int_{-C^z}^{C^z} \xi_*\partial_\xi\Omega_* \,\mathrm{d}\xi_*} = \int_{-C^z}^{C^z} (\xi_* - f)\partial_\xi\Omega_* \,\mathrm{d}\xi_* = 0.$$

We also have the following property.

Lemma 6.6 (Spine approximation property)**.** *Let θ be a C^2 almost-sharp front, and let S be the spine curve defined by the above (6.5). Then for any $\Gamma = \Gamma(x) \in C_c^2(\mathbb{R}^2)$, as $\delta \to 0$,*

$$\int_{x\in\mathbb{R}^2} \Gamma\nabla^\perp\theta \,\mathrm{d}x = -\int_{\mathbb{T}} \Gamma(S_*)\partial_s S_* \,\mathrm{d}s_* + O(\|\Gamma\|_{C^2}\delta^2). \tag{6.4}$$

That is, when testing against functions $\Gamma \in C_c^2(\mathbb{R}^2)$, we have the approximation of the vector-valued measure,

$$\nabla^\perp\theta \,\mathrm{d}x = -\partial_s S_* \mathrm{d}\delta_{\xi_*=f_*} \,\mathrm{d}s_* + O(\delta^2)$$

in the tubular neighbourhood. We remark that in (6.4), we keep the $\|\Gamma\|_{C^2}$ dependence in the $O(\delta^2)$ constant because we will require the use of test functions with $\|\Gamma\|_{C^2}$ that degenerate as $\delta \to 0$ (specifically in understanding (6.13)).

Proof. Write $\tilde{\Gamma} = \Gamma(x(s,\xi))$, with $x = z + \delta\xi N$. We are looking for the following expansion

$$\iint \left(\delta\partial_s\Omega_* N_* - L(1-\delta\kappa_*\xi_*)\partial_\xi\Omega_* T_*\right)\tilde{\Gamma}_*\,\mathrm{d}s_*\,\mathrm{d}\xi_* \\ + \int_{\mathbb{T}} \tilde{\Gamma}_*\partial_s S_*\,\mathrm{d}s_* \stackrel{?}{=} O(\delta^2). \tag{6.5}$$

We Taylor expand around S_* in the direction of N_* to obtain the following asymptotic:

$$\begin{aligned}\tilde{\Gamma}_* = \Gamma(x_*) &= \Gamma(S_*) + \nabla\Gamma(S_*)\cdot(x_* - S_*) + O(|x_* - S_*|^2)\\ &= \Gamma(S_*) + \delta\nabla\Gamma(S_*)\cdot N_*(\xi_* - f_*) + O(\delta^2).\end{aligned}$$

Plugging into the left hand side (LHS) of (6.5) and collecting coefficients of $\Gamma(S)$ and $\nabla\Gamma(S)$,

$$\begin{aligned}&\text{LHS}\\ &= \int_{\mathbb{T}}\Bigg[\Gamma(S_*)\left(\partial_s S_* - T_*\int_{-1}^{1} L(1-\delta\kappa_*\xi_*)\partial_\xi\Omega_*\,\mathrm{d}\xi_* + \delta N_*\int_{-1}^{1}\partial_s\Omega_*\,\mathrm{d}\xi_*\right)\\ &+ \delta\nabla\Gamma(S_*)\cdot N_*\int_{-1}^{1}(\xi_* - f_*)\left(\delta\partial_s\Omega_* N_* - L(1-\delta\kappa_*\xi_*)\partial_\xi\Omega_* T_*\right)\,\mathrm{d}\xi_*\Bigg]\,\mathrm{d}s_*\\ &+ O(\delta^2)\\ &= \int_{\mathbb{T}}\Bigg[\Gamma(S_*)\left(\partial_s S_* - T_*\int_{-1}^{1} L(1-\delta\kappa_*\xi_*)\partial_\xi\Omega_*\,\mathrm{d}\xi_* + \delta N_*\int_{-1}^{1}\partial_s\Omega_*\,\mathrm{d}\xi_*\right)\\ &- L\delta\nabla\Gamma(S_*)\cdot N_*\int_{-1}^{1}(\xi_* - f_*)\partial_\xi\Omega_* T_*\,\mathrm{d}\xi_*\Bigg]\,\mathrm{d}s_* + O(\delta^2).\end{aligned} \tag{6.6}$$

The following identities follow immediately from the definitions of S and f,

$$\begin{aligned}0 &= f_*' + \int_{-1}^{1}\partial_s\Omega_*\,\mathrm{d}\xi,\\ 0 &= \int_{-1}^{1}(\xi_* - f_*)\partial_\xi\Omega_*\,\mathrm{d}\xi, \text{ and}\\ \partial_s S_* &= L(1-\delta\kappa_* f_*)T_* + \delta f_*' N_*\\ &= L(1-\delta\kappa_* f_*)T_*\int_{-1}^{1}\partial_\xi\Omega_*\,\mathrm{d}\xi + \delta f_*' N_*,\end{aligned}$$

showing that the right-hand side of (6.6) is of order δ^2, as the terms in the square brackets vanish. $\square$

6.2.1 Evolution of a spine

Proposition 6.2 showed that any compatible curve evolves by the sharp front equation (2.3) up to an error of order $O(\delta^{1-\alpha})$. However, for the spine, we will be able to improve this to the better error rate $O(\delta^{2-\alpha})$. This shows that in a sense, the spine curve arises as the correct correction at length scales $\sim \delta$ of a compatible curve.

Theorem 6.7. *For an ASF solution to (1.1), the spine curve S defined above in Definition 6.5 evolves (in the weak sense) according to the sharp front equation up to $O(\delta^{2-\alpha})$ error. That is,*

$$\partial_t S \cdot N = \left(- \int_{s_* \in I} K(S - S_*)(\partial_s S_* - \partial_s S)\, \mathrm{d}s_* \right) \cdot N + O(\delta^{2-\alpha}). \tag{6.7}$$

Proof. Without loss of generality, suppose that the constant for the base curve $C^z = 1$, and let us choose $\delta \ll 1$ so that we can extend the ξ coordinate to the range $[-3, 3]$. This amounts to having a well defined neighbourhood of thickness 6δ. The constants chosen here are arbitrary and only serve to simplify the later computations. Recall that θ is a weak solution if for every $\Gamma \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$,

$$0 = \iint_{t \geq 0, x \in \mathbb{R}^2} \theta \partial_t \Gamma + \theta (u \cdot \nabla \Gamma)\, \mathrm{d}x\, \mathrm{d}t. \tag{6.8}$$

Define for each time t the spine curve $S = S(s,t) \in A_{\text{in}}^S(t)$, the inner region bounded by the closed curve $S + 2\delta N$ with N the inward normal, the outer region $A_{\text{out}}^S(t)$ bounded by $S - 2\delta N$, and the tubular region $A_{\text{mid}}^S(t)$ in the middle of radius 2δ. We give the names S^+ and S^- to the inner and outer boundary curves of A_{mid}^S respectively,

$$\begin{aligned} S^+ &:= S + 2\delta N = z + \delta(f + 2)N, \\ S^- &:= S - 2\delta N = z + \delta(f - 2)N. \end{aligned}$$

In addition, we will use the mildly abusive notation

$$S^\sigma = \begin{cases} S^+ & \sigma = +1, \\ S^- & \sigma = -1. \end{cases}$$

We thus have for each t (up to null sets),

$$\begin{aligned}\mathbb{R}^2 &= A^S_{\text{in}}(t) \cup A^S_{\text{out}}(t) \cup A^S_{\text{mid}}(t),\\ A^S_{\text{in}}(t) &\subseteq \{x : \theta(x,t) = +1/2\},\\ A^S_{\text{out}}(t) &\subseteq \{x : \theta(x,t) = -1/2\}.\end{aligned}$$

Also define the related partition of $\mathbb{R}^2 \times [0,\infty)$ by

$$\begin{aligned}A^S_{\text{in}} &= \bigcup_{t\geq 0} A^S_{\text{in}}(t) \times \{t\}\,,\\ A^S_{\text{out}} &= \bigcup_{t\geq 0} A^S_{\text{out}}(t) \times \{t\}\,,\\ A^S_{\text{mid}} &= \bigcup_{t\geq 0} A^S_{\text{mid}}(t) \times \{t\}\,.\end{aligned}$$

We treat the two integrands $\theta\partial_t\Gamma$ and $\theta(u\cdot\nabla\Gamma)$ in (6.8) separately, with the three sets to integrate over. For the first integrand, we have the three terms,

$$\left(\iint_{A^S_{\text{in}}} + \iint_{A^S_{\text{out}}} + \iint_{A^S_{\text{mid}}}\right)\theta\partial_t\Gamma\,\mathrm{d}x\,\mathrm{d}t =: I_{\text{in}} + I_{\text{out}} + I_{\text{mid}}.$$

For legibility reasons, we will abusively write

$$\text{in} := A^S_{\text{in}}, \quad \text{mid} := A^S_{\text{mid}}, \quad \text{out} := A^S_{\text{out}}.$$

In the tubular coordinates around z, with $L_1(s,\xi) = L(1-\delta\kappa(s)\xi) = L + O(\delta)$,

$$\begin{aligned}I_{\text{mid}} &= \iint_{t\geq 0, s\in\mathbb{T}} \int_{\xi=f-2}^{f+2} \Omega(s,\xi)\partial_t\Gamma(x(s,\xi))\delta L_1(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t\\ &= \iint_{t\geq 0, s\in\mathbb{T}} \int_{\xi=f-2}^{f+2} \Omega(s,\xi,t)\partial_t\Gamma(x(s,\xi),t)L\delta\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t + O(\delta^2).\end{aligned}$$

By the spine cancellation property (6.3) we have

$$\begin{aligned}&I_{\text{mid}}\\ &= \delta\iint_{t\geq 0, s\in\mathbb{T}} \int_{\xi=f-2}^{f+2} \Omega[\partial_t\Gamma(x(s,\xi),t) - \partial_t\Gamma(S(s),t)]L\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t + O(\delta^2)\\ &= O(\delta^2),\end{aligned}$$

since $|\partial_t\Gamma(x(s,\xi),t) - \partial_t\Gamma(S(s),t)| \lesssim |x(s,\xi) - S(s)| = O(\delta)$ uniformly in s and ξ.

For I_{in}, we apply the Divergence Theorem in 3D,

$$\begin{aligned}\iint_{(x,t)\in\text{in}} \theta\partial_t\Gamma\,\mathrm{d}t\,\mathrm{d}x &= \frac{1}{2}\iint_{(x,t)\in\text{in}} \nabla_{x^1,x^2,t}\cdot\begin{bmatrix}0\\0\\\Gamma\end{bmatrix}\mathrm{d}t\,\mathrm{d}x\\ &= \frac{1}{2}\iint_{t\geq 0,s\in\mathbb{T}} \Gamma(S^+,t)\partial_t[S^+]\cdot\partial_s[S^+]^\perp\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Similarly for I_{out} we obtain the term (note the minus sign from the opposite orientation)

$$\begin{aligned}\iint_{(x,t)\in\text{out}} \theta\partial_t\Gamma\,\mathrm{d}t\,\mathrm{d}x &= \frac{-1}{2}\iint_{t\geq 0,s\in\mathbb{T}} \Gamma(S^-,t)(-\partial_t[S^-])\cdot\partial_s[S^-]^\perp\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{2}\iint_{t\geq 0,s\in\mathbb{T}} \Gamma(S^-,t)\partial_t[S^-]\cdot\partial_s[S^-]^\perp\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Since $S^\pm = S \pm O(\delta)$, we obtain by the approximation formula valid for C^2 functions,

$$f(a+b)+f(a-b) = 2f(a) + O(b^2), \tag{6.9}$$

(with the constant implicit in the $O(b^2)$ notation depending on $\|f''\|_{L^\infty}$) that

$$\iint_{\text{in}\cup\text{out}} \theta\partial_t\Gamma\,\mathrm{d}t\,\mathrm{d}x = \iint_{t\geq 0,s\in\mathbb{T}} \Gamma(S)\partial_t S\cdot\partial_s S^\perp\,\mathrm{d}s\,\mathrm{d}t + O(\delta^2),$$

with implicit constant depending on $\|\Gamma\|_{C^2}$ and the geometry of the base curve z. We therefore obtain that the first term is

$$\begin{aligned}I_{\text{in}} + I_{\text{out}} + I_{\text{mid}} &= \iint_{t\geq 0,x\in\mathbb{R}^2} \theta\partial_t\Gamma\,\mathrm{d}t\,\mathrm{d}x\\ &= \iint_{t\geq 0,s\in\mathbb{T}} \Gamma(S)\partial_t S\cdot\partial_s S^\perp\,\mathrm{d}s\,\mathrm{d}t + O(\delta^2).\end{aligned}$$

For the second term, define $B(t)$ as the following integrand,

$$\int_{x\in\mathbb{R}^2,t\geq 0} \theta u\cdot\nabla\Gamma\,\mathrm{d}x\,\mathrm{d}t =: \int_{t\geq 0} B(t)\,\mathrm{d}t.$$

We need to control $B(t)$, which has compact support in t due to Γ. There will be no interaction with t so we suppress the time dependence in the integrands. We now symmetrise the integrand of $B(t)$ by reversing the roles of x and y, as follows:

$$B(t) = \int_{x\in\mathbb{R}^2}\int_{y\in\mathbb{R}^2} \nabla_x^\perp(|x-y|^{-1-\alpha})\cdot\nabla\Gamma(x)\theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y, \text{ and}$$

$$\begin{aligned} B(t) &= \int_{x\in\mathbb{R}^2}\int_{y\in\mathbb{R}^2} \nabla_y^\perp(|x-y|^{-1-\alpha})\cdot\nabla\Gamma(y)\theta(y)\theta(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{x\in\mathbb{R}^2}\int_{y\in\mathbb{R}^2} -\nabla_x^\perp(|x-y|^{-1-\alpha})\cdot\nabla\Gamma(y)\theta(y)\theta(x)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which implies that

$$2B(t) = \int_{x\in\mathbb{R}^2}\int_{y\in\mathbb{R}^2} \nabla_x^\perp|x-y|^{-1-\alpha}\cdot[\nabla\Gamma(x)-\nabla\Gamma(y)]\theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y.$$

We split $\mathbb{R}^2\times\mathbb{R}^2 = C_0\cup C_1\cup C_2$, where the subscript in C_i depends on whether none of, one of, or both of x and y are in mid respectively,

$$\begin{aligned} C_0 &= (\mathrm{mid})^c\times(\mathrm{mid})^c, \\ C_1 &= [\mathrm{mid}\times(\mathrm{mid})^c]\cup[(\mathrm{mid})^c\times\mathrm{mid}], \\ C_2 &= \mathrm{mid}\times\mathrm{mid}. \end{aligned}$$

Then define B_0, B_1, B_2 by

$$B_i(t) = \frac{1}{2}\iint_{(x,y)\in C_i} \nabla_x^\perp|x-y|^{-1-\alpha}\cdot[\nabla\Gamma(x)-\nabla\Gamma(y)]\theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y,$$

so that $B_0(t)+B_1(t)+B_2(t) = B(t)$. These terms differ based on if θ is locally constant on C_i. For instance, if $\theta(x) = \pm\frac{1}{2}$ then integration by parts in x is simpler than integration by parts in y, and is likely to simplify the calculation.

B_1 is an error term

We show this by integrating by parts in y, which gives a difference of two line integrals (corresponding to the two boundary components of ∂mid.) First, we separate B_1 into two similar terms:

$$\begin{aligned} 2B_1(t) &= \iint_{(x,y)\in C_1} \nabla_x^\perp|x-y|^{-1-\alpha}\cdot[\nabla\Gamma(x)-\nabla\Gamma(y)]\theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \left(\iint_{x\in\mathrm{mid},y\in(\mathrm{mid})^c} + \iint_{x\in(\mathrm{mid})^c,y\in\mathrm{mid}}\right)\nabla_x^\perp|x-y|^{-1-\alpha}\cdot \\ &\quad [\nabla\Gamma(x)-\nabla\Gamma(y)]\theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y \\ &=: B_{11}+B_{12}. \end{aligned}$$

We show below that B_{12} is of order $O(\delta^{2-\alpha})$; B_1 can be estimated in the same way. After integrating by parts the derivative in $\nabla_x^\perp|x-y|^{-1-\alpha}$ and using the product rule

$\nabla^\perp \cdot (fV) = \nabla^\perp f \cdot V + f\nabla^\perp \cdot V$, we are left with only the boundary terms because $\nabla^\perp \cdot \nabla\Gamma = 0$ and also $\nabla^\perp\theta|_{x\in(\mathrm{mid})^c} = 0$. For arbitrary vector valued functions F, note that the Divergence Theorem gives

$$\begin{aligned}
&\int_{\mathrm{mid}^c} \nabla \cdot F(x)\,\mathrm{d}x \\
&= \int_{\partial\mathrm{mid}^c} F(x) \cdot N_{\mathrm{out},\partial\mathrm{mid}^c}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{T}} F(S_*^-) \cdot (\partial_s S_*^-)^\perp\,\mathrm{d}s_* - \int_{\mathbb{T}} F(S_*^+) \cdot (\partial_s S_*^+)^\perp\,\mathrm{d}s_* \\
&= -\sum_{\sigma=\pm1} \int_{\mathbb{T}} F(S_*^\sigma) \cdot (\partial_s S_*^\sigma)^\perp \sigma\,\mathrm{d}s_*.
\end{aligned}$$

So as $\theta(S^\sigma) = \frac{\sigma}{2}$, $\nabla^\perp \cdot F = \nabla \cdot (-F^\perp)$, and $\nabla\theta|_{\partial\mathrm{mid}} = 0$, we have the following special cases of the Divergence Theorem, which we will repeatedly use:

$$\int_{\mathrm{mid}^c} \nabla^\perp \cdot F(x)\theta(x)\,\mathrm{d}x = \frac{1}{2}\sum_{\sigma=\pm1}\int_{\mathbb{T}} F(S_*^\sigma) \cdot (\partial_s S_*^\sigma)\,\mathrm{d}s_*, \tag{6.10}$$

$$\int_{\mathrm{mid}^c} \nabla \cdot F(x)\theta(x)\,\mathrm{d}x = -\frac{1}{2}\sum_{\sigma=\pm1}\int_{\mathbb{T}} F(S_*^\sigma) \cdot (\partial_s S_*^\sigma)^\perp\,\mathrm{d}s_*. \tag{6.11}$$

Therefore, we can write B_{12} as follows:

$$\begin{aligned}
&B_{12} \\
&= \int_{\mathrm{mid}} \theta(y) \int_{\partial\mathrm{mid}^c} |x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)] \cdot T_{\partial\mathrm{mid}^c}(x)\theta(x)\,\mathrm{d}l(x)\,\mathrm{d}y \\
&= \int_{\mathrm{mid}} \frac{\theta(y)}{2} \underbrace{\sum_{\sigma=\pm1}\int_{\mathbb{T}} |S_*^\sigma - y|^{-1-\alpha}[\nabla\Gamma(S_*^\sigma) - \nabla\Gamma(y)] \cdot \partial_s S_*^\sigma \;\mathrm{d}s_*}_{=:G(y)}\,\mathrm{d}y.
\end{aligned}$$

Writing the y-integral in tubular coordinates around S, we can use the cancellation identity (6.3) to see that

$$2B_{12} = \int_{\mathbb{T}} \int_{\xi=f-2}^{f+2} \Omega(s,\xi)(G(y(s,\xi)) - G(S))\delta L\,\mathrm{d}s\,\mathrm{d}\xi + O(\delta^2).$$

In what follows, we will write y for the parameterised point $y = y(s,\xi)$. If we can prove $|G(y) - G(S)| \lesssim \delta^{1-\alpha}$, this would imply that $B_{12} = O(\delta^{2-\alpha})$. We now use a smooth cut-off function $\rho_\delta(s) = \rho(s/\delta)$ with $\operatorname{supp}\rho = [-1,1]$, $\rho|_{[-1/2,1/2]} = 1$ to split the $G(y)$ into two parts,

$$
\begin{aligned}
G(y) = &\sum_{\sigma=\pm1}\int_{\mathbb{T}} \underbrace{|S_*^\sigma - y|^{-1-\alpha}[\nabla\Gamma(S_*^\sigma) - \nabla\Gamma(y)]}_{\mathcal{G}(S_*^\sigma, y)} \cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
=: &\sum_{\sigma=\pm1}\int_{\mathbb{T}} \mathcal{G}(S_*^\sigma, y)\cdot \partial_{s_*} S_*^\sigma\,\mathrm{d}s_* \\
= &\sum_{\sigma=\pm1}\int_{\mathbb{T}} \rho_\delta(s-s_*)\mathcal{G}(S_*^\sigma, y)\cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
&+ \sum_{\sigma=\pm1}\int_{\mathbb{T}} (1-\rho_\delta(s-s_*))\,\mathcal{G}(S_*^\sigma, y)\cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
=: &\, G_1(y) + G_2(y).
\end{aligned}
$$

Note the function $\mathcal{G} = \mathcal{G}(x,y)$ defined for notational convenience by the above lines, with $x = S_*^\sigma$. For G_1, the support in s_* of $\rho_\delta(s-s_*)$ gives us the required control using $|\mathcal{G}(x,y)| \lesssim_\Gamma |x-y|^{-\alpha}$, so that $|\mathcal{G}(S_*^\sigma, y)| = O(|s_* - s|^{-\alpha})$. Therefore, we have the bound

$$
|G_1| \le \|\rho\|_{L^\infty}\|\partial_s S\|_{L^\infty}\|\mathcal{G}(S_*^\sigma, y)\|_{L^1_{s_*}[s-\delta, s+\delta]} = O_{\rho,\Gamma}(\delta^{1-\alpha}).
$$

So it now suffices to study the derivative of G_2, since

$$
\begin{aligned}
|G(y) - G(S)| &\lesssim |G_2(y) - G_2(S)| + O(\delta^{1-\alpha}) \\
&\lesssim \|\nabla G_2\|_{L^\infty}|y - S| + O(\delta^{1-\alpha}) \\
&= \|\nabla G_2\|_{L^\infty} O(\delta) + O(\delta^{1-\alpha}).
\end{aligned}
$$

In the tubular coordinates $y = y(s,\xi)$, we have to control the two terms $\partial_s G_2(y(s,\xi))$ and $\partial_\xi G_2(y(s,\xi))/\delta$. The first term is

$$
\begin{aligned}
&\partial_s G_2(y) \\
&= \sum_{\sigma=\pm1}\int_{\mathbb{T}} -\partial_{s_*}(1-\rho_\delta(s-s_*))\,\mathcal{G}(S_*^\sigma, y)\cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
&+ \sum_{\sigma=\pm1}\int_{\mathbb{T}} (1-\rho_\delta(s-s_*))\,\partial_s\mathcal{G}(S_*^\sigma, y)\cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
&= \sum_{\sigma=\pm1}\int_{\mathbb{T}} (1-\rho_\delta(s-s_*))\,(\partial_{s_*}\mathcal{G}(S_*^\sigma, y) + \partial_s\mathcal{G}(S_*^\sigma, y))\cdot \partial_{s_*}(S_*^\sigma)\,\mathrm{d}s_* \\
&+ \sum_{\sigma=\pm1}\int_{\mathbb{T}} (1-\rho_\delta(s-s_*))\,\mathcal{G}(S_*^\sigma, y)\cdot \partial^2_{s_*}(S_*^\sigma)\,\mathrm{d}s_*.
\end{aligned}
$$

The worst term $\partial_{s_*}\mathcal{G}(S^\sigma_*, y) + \partial_s \mathcal{G}(S^\sigma_*, y)$ is $O(|s_* - s|^{-1-\alpha})$, and the cutoff function restricts the integration to the region $|s - s_*| \geq \delta/2$. Thus, $\partial_s G_2 = O(\delta^{-\alpha})$.

The other term $\frac{1}{\delta}\partial_\xi G_2(y)$ is simpler since the cutoff ρ_δ does not depend on ξ,

$$\begin{aligned} \frac{1}{\delta}\partial_\xi G_2(y) = \sum_{\sigma=\pm1} \int_{\mathbb{T}} (1 - \rho_\delta(s - s_*)) \times \\ \times \quad \underbrace{\frac{1}{\delta}\partial_\xi \left(\frac{\nabla\Gamma(S^\sigma_*) - \nabla\Gamma(y)}{|S^\sigma_* - y|^{1+\alpha}} \right)}_{=:G_{2,1}(y)} \cdot \partial_{s_*}(S^\sigma_*)\, \mathrm{d}s_* \end{aligned}$$

Note that $\frac{1}{\delta}\partial_\xi G_{2,1}(y) = \nabla_y G_{2,1}(y) \cdot \frac{1}{\delta}\partial_\xi y$, and $\frac{1}{\delta}\partial_\xi y = N = O(1)$; so we have an $O(|s - s_*|^{-1-\alpha})$ integrand, integrated on the region $|s - s_*| > \delta/2$. Hence, we see that $G_2(y) = O(\delta^{2-\alpha})$, so $B_{11}, B_{12} = O(\delta^{2-\alpha})$, and therefore

$$B_1 = O(\delta^{2-\alpha}).$$

B_2 is an error term

Here, we are forced to use integration by parts instead of (merely) the Divergence Theorem, since neither $\theta(x)$ or $\theta(y)$ is locally constant. This leaves us with an integration against $\nabla^\perp\theta\, \mathrm{d}x$, allowing the use of Lemma 6.6:

$$\begin{aligned} &2B_2(t) \\ &= \iint_{(x,y)\in C_2} \nabla^\perp_x |x - y|^{-1-\alpha} \cdot [\nabla\Gamma(x) - \nabla\Gamma(y)]\theta(x)\theta(y)\, \mathrm{d}x\, \mathrm{d}y \\ &= -\frac{1}{2}\sum_{\sigma=\pm1} \int_{\mathbb{T}} \int_{\text{mid}} \theta(y)|S^\sigma_* - y|^{-1-\alpha}[\nabla\Gamma(S^\sigma_*) - \nabla\Gamma(y)] \cdot \partial_{s_*}(S^\sigma_*)\, \mathrm{d}s_*\, \mathrm{d}y \quad (6.12) \\ &\quad - \int_{\text{mid}} \underbrace{\left(\int_{\text{mid}} |x - y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\theta(y)\, \mathrm{d}y \right)}_{=:Q(x)} \cdot \nabla^\perp\theta(x)\, \mathrm{d}x. \quad (6.13) \end{aligned}$$

Above, the $\nabla^\perp_x$ never falls on $\nabla\Gamma$ due to $\nabla \cdot \nabla^\perp = 0$. For (6.12), the singularity is no worse than the one for B_1 and can be treated in exactly the same way. For (6.13), we aim to use (6.4) of Lemma 6.6, so we need to estimate $\|\nabla^2 Q\|_{C^2}$. In what follows, we concatenate vectors to denote a tensor e.g. $(UWV)_{ijk} = U_i W_j V_k$, and $(\nabla^2 F)_{ijk} = \partial_i \partial_j F_k$. Then integrating by parts twice introduces the term $r(x) = \int_{\text{mid}} (\nabla^2_x - \nabla^2_y)\left(|x - y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right)\theta(y)\, \mathrm{d}y$:

$$\nabla^2 Q(x) = \int_{\text{mid}} \nabla^2_x \left(|x - y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right)\theta(y)\, \mathrm{d}y$$

$$
\begin{aligned}
&= \int_{\mathrm{mid}} \nabla_y^2 \left(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right) \theta(y)\,\mathrm{d}y + r(x) && (6.14)\\
&= \int_{\partial\mathrm{mid}} \nabla_y \left(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right) \theta(y) N_{\mathrm{out},\partial\mathrm{mid}}\,\mathrm{d}l(y) && (6.15)\\
&+ \int_{\mathrm{mid}} \nabla_y (|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]) \nabla\theta(y)\,\mathrm{d}y + r(x). && (6.16)
\end{aligned}
$$

The term $r(x) = \int_{\mathrm{mid}} R(x,y)\theta(y)\,\mathrm{d}y$ is given explicitly by

$$
\begin{aligned}
R(x,y) &= (\nabla_x^2 - \nabla_y^2)\left(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right)\\
&= |x-y|^{-1-\alpha}(\nabla^3\Gamma(x) + \nabla^3\Gamma(y))\\
&+ (-1-\alpha)\frac{x-y}{|x-y|^{3+\alpha}}(\nabla^2\Gamma(x) - \nabla^2\Gamma(y))\\
&+ (-1-\alpha)(\nabla^2\Gamma(x) - \nabla^2\Gamma(y))\frac{x-y}{|x-y|^{3+\alpha}}\\
&= O(|x-y|^{-1-\alpha}).
\end{aligned}
$$

The boundary term (6.15) from integration by parts is

$$
\begin{aligned}
&\int_{\partial\mathrm{mid}} \nabla_y \left(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right) \theta(y) N_{\mathrm{out},\partial\mathrm{mid}}\,\mathrm{d}s\\
&= \frac{1}{2}\sum_{\sigma=\pm1} \int_{\mathbb{T}} \nabla_y \left(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)]\right)\Big|_{y=S^\sigma} (\partial_s S^\sigma)^\perp\,\mathrm{d}s.
\end{aligned}
$$

For the remaining term in (6.16), we have

$$
\begin{aligned}
&|\nabla_y(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)])|\\
&= \left|\frac{(-1-\alpha)(x-y)(\nabla\Gamma(x) - \nabla\Gamma(y)) + \nabla^2\Gamma(y)|x-y|^2}{|x-y|^{3+\alpha}}\right|\\
&= \frac{(1+\alpha)|\nabla^2\Gamma(x)|}{|x-y|^{1+\alpha}} + O(|x-y|^{-\alpha}).
\end{aligned}
$$

All of these terms (6.14), (6.15), (6.16) are $O(\delta^{-\alpha})$ terms, which can be seen by using the asymptotic lemma 5.8 to compute the s integral to leading order. For instance, for (6.16), we write out the integral explicitly using the coordinates $y = z(s_*) + \delta\xi_* N(s_*)$, $x = z(s) + \delta\xi N(s)$, and the growth condition $|\nabla\theta| \lesssim \frac{1}{\delta}$ from (5.1),

$$
\begin{aligned}
&\left|\int_{\mathrm{mid}} \nabla_y(|x-y|^{-1-\alpha}[\nabla\Gamma(x) - \nabla\Gamma(y)])\nabla\theta(y)\,\mathrm{d}y\right|\\
&\le \frac{1}{\delta}(1+\alpha)|\nabla^2\Gamma(x)| \int_{\xi=f-2}^{f+2} \int_{\mathbb{T}} \frac{\delta L(1-\delta\kappa_*\xi_*)}{|z - z_* + \delta(\xi N - \xi_* N_*)|^{1+\alpha}}\,\mathrm{d}s_*\,\mathrm{d}\xi_*
\end{aligned}
$$

$$= O(\delta^{-\alpha}),$$

which follows by applying the asymptotic Lemma 5.8. Thus, $\|\nabla^2 Q\|_{L^\infty} = O(\delta^{-\alpha})$, and Lemma 6.6 implies that

$$\begin{aligned}2B_2(t) = &\int_{\mathbb{T}} \left(\int_{\text{mid}} |S-y|^{-1-\alpha} [\nabla\Gamma(S) - \nabla\Gamma(y)] \theta(y) \,\mathrm{d}y \right) \cdot \partial_s S \,\mathrm{d}s \\ &+ O(\delta^{2-\alpha}).\end{aligned}$$

Reversing the order of integration and using the tubular coordinates we notice we can again use the spine condition (6.3) to bring out an extra cancellation. That is, defining H by

$$H(y_*) := \int_{\mathbb{T}} |S-y_*|^{-1-\alpha} [\nabla\Gamma(S) - \nabla\Gamma(y_*)] \cdot \partial_s S \,\mathrm{d}s,$$

we deduce that

$$\begin{aligned}2B_2(t) &= \int_{f-2}^{f+2} \int_{\mathbb{T}} H(y_*) \Omega_* \delta L_{1*} \,\mathrm{d}s_* \,\mathrm{d}\xi_* + O(\delta^{2-\alpha}) \\ &= \delta \int_{f-2}^{f+2} \int_{\mathbb{T}} \Omega_* H(y_*) L \,\mathrm{d}s_* \,\mathrm{d}\xi_* + O(\delta^{2-\alpha}) \\ &= \delta \int_{f-2}^{f+2} \int_{\mathbb{T}} \Omega_* \left(H(y_*) - H(S_*) \right) L \,\mathrm{d}s_* \,\mathrm{d}\xi_* + O(\delta^{2-\alpha}) \\ &= O(\delta^{2-\alpha}).\end{aligned}$$

B_0 is the evolution term plus an error

The final term to estimate is

$$B_0(t) = \int_{(\text{mid})^c} \int_{(\text{mid})^c} \nabla_x^\perp |x-y|^{-1-\alpha} \cdot \nabla\Gamma(x) \theta(x) \theta(y) \,\mathrm{d}x \,\mathrm{d}y.$$

We use both gradients appearing in B_0 to integrate by parts (via the formulas (6.11),(6.10)) , on which we obtain only boundary terms due to either of the two cancellations $\nabla^\perp \cdot \nabla = 0$ or $\nabla\theta|_{\partial\text{mid}} = 0$:

$$\begin{aligned}&B_0(t) \\ &= \int_{(\text{mid})^c} \int_{(\text{mid})^c} \nabla_x^\perp |x-y|^{-1-\alpha} \cdot \nabla\Gamma(x) \theta(x) \theta(y) \,\mathrm{d}x \,\mathrm{d}y \\ &= -\frac{1}{2} \sum_{\sigma_1 = \pm 1} \int_{(\text{mid})^c} \int_{\mathbb{T}} \nabla_x^\perp |x-y|^{-1-\alpha}|_{x=S^{\sigma_1}} \cdot (\partial_s S^{\sigma_1})^\perp \Gamma(S^{\sigma_1}) \theta(y) \,\mathrm{d}s \,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{\sigma_1=\pm 1} \int_{(\mathrm{mid})^c} \int_{\mathbb{T}} \nabla_y^\perp |S^{\sigma_1} - y|^{-1-\alpha} \cdot (\partial_s S^{\sigma_1})^\perp \Gamma(S^{\sigma_1}) \theta(y) \, \mathrm{d}s \, \mathrm{d}y \\
&= \frac{1}{4} \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma_1} - S_*^{\sigma_2}|^{-1-\alpha} \cdot (\partial_s S^{\sigma_1})^\perp \cdot \partial_s S_*^{\sigma_2} \Gamma(S^{\sigma_1}) \, \mathrm{d}s \, \mathrm{d}s_* \\
&= -\frac{1}{4} \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma_1} - S_*^{\sigma_2}|^{-1-\alpha} \partial_s S^{\sigma_1} \cdot [\partial_s S_*^{\sigma_2}]^\perp \Gamma(S^{\sigma_1}) \, \mathrm{d}s \, \mathrm{d}s_*.
\end{aligned}
$$

This sum of four terms will now be grouped into two terms, one where $\sigma_1 = \sigma_2$ and one where $\sigma_1 = -\sigma_2$,

$$
\begin{aligned}
B_0(t) &= B_{00}(t) + B_{01}(t), \\
B_{00}(t) &= -\frac{1}{4} \sum_{\substack{\sigma_1, \sigma_2 = \pm 1 \\ \sigma_1 = \sigma_2}} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma_1} - [S^{\sigma_2}]_*|^{-1-\alpha} \partial_s S^{\sigma_1} \cdot [\partial_s S^{\sigma_2}]_*^\perp \Gamma(S^{\sigma_1}) \, \mathrm{d}s \, \mathrm{d}s_* \\
&= -\frac{1}{4} \sum_{\sigma=\pm 1} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma} - [S^{\sigma}]_*|^{-1-\alpha} \partial_s S^{\sigma} \cdot [\partial_s S^{\sigma}]_*^\perp \Gamma(S^{\sigma}) \, \mathrm{d}s \, \mathrm{d}s_*, \\
B_{01}(t) &= -\frac{1}{4} \sum_{\substack{\sigma_1, \sigma_2 = \pm 1 \\ \sigma_1 \neq \sigma_2}} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma_1} - [S^{\sigma_2}]_*|^{-1-\alpha} \partial_s S^{\sigma_1} \cdot [\partial_s S^{\sigma_2}]_*^\perp \Gamma(S^{\sigma_1}) \, \mathrm{d}s \, \mathrm{d}s_* \\
&= -\frac{1}{4} \sum_{\sigma=\pm 1} \int_{\mathbb{T}} \int_{\mathbb{T}} |S^{\sigma} - [S^{-\sigma}]_*|^{-1-\alpha} \partial_s S^{\sigma} \cdot [\partial_s S^{-\sigma}]_*^\perp \Gamma(S^{\sigma}) \, \mathrm{d}s \, \mathrm{d}s_*.
\end{aligned}
$$

For $B_{00}(t)$, we can use the formula (6.9) to get that

$$
B_{00}(t) = -\frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_s S \cdot \partial_s S_*^\perp}{|S - S_*|^{1+\alpha}} \Gamma(S) \, \mathrm{d}s \, \mathrm{d}s_* + O(\delta^2).
$$

Treating $B_{01}(t) =: \mathcal{B}(\delta)$ as a function of δ, we need to prove that

$$
\mathcal{B}(\delta) = \mathcal{B}(0) + O(\delta^{2-\alpha}),
$$

since $\mathcal{B}(0) = -\frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_s S \cdot \partial_s S_*^\perp}{|S - S_*|^{1+\alpha}} \Gamma(S) \, \mathrm{d}s \, \mathrm{d}s_*$ with $B_{00}(t)$ gives the required evolution term of the spine:

$$
B_{00}(t) + \mathcal{B}(0) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \frac{\partial_s S_* - \partial_s S}{|S - S_*|^{1+\alpha}} \, \mathrm{d}s_* \right) \cdot \partial_s S^\perp \Gamma(S) \, \mathrm{d}s + O(\delta^2).
$$

Symmetrizing as before, we obtain

$$
\mathcal{B}(\delta) = -\frac{1}{8} \sum_{\sigma=\pm 1} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_s S^\sigma \cdot [\partial_s S^{-\sigma}]_*^\perp}{|S^\sigma - [S^{-\sigma}]_*|^{1+\alpha}} \left(\Gamma(S^\sigma) - \Gamma([S^{-\sigma}]_*) \right) \mathrm{d}s \, \mathrm{d}s_*.
$$

Since the claimed result (6.7) is a result about test functions supported on the curve S, we may assume that Γ is constant along the normal N on a δ neighbourhood of S, giving

$$\mathcal{B}(\delta) = -\frac{1}{8}\sum_{\sigma=\pm 1}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\partial_s S^\sigma \cdot [\partial_s S^{-\sigma}]_*^\perp}{|S^\sigma - [S^{-\sigma}]_*|^{1+\alpha}} \left(\Gamma(S) - \Gamma(S_*)\right) \mathrm{d}s\, \mathrm{d}s_*.$$

Note that $\mathcal{B}(0)$ has the well-behaved $O(|s-s_*|^{1-\alpha})$ integrand. Recalling that $\mathcal{B}'(\tilde{\delta}) = \frac{\mathcal{B}(\delta)-\mathcal{B}(0)}{\delta}$ for some $\tilde{\delta} \in (0,\delta)$, it suffices to prove that

$$\mathcal{B}'(\delta) \stackrel{?}{=} O(\delta^{1-\alpha}),$$

since $s \mapsto s^{1-\alpha}$ is increasing for $0 < s$. On differentiation with respect to δ, a factor of σ appears, which means the sum over $\sigma = \pm 1$ becomes a symmetric difference. We expand the shorthand notation $\partial_s S^\sigma \cdot [\partial_s S^{-\sigma}]_*^\perp$ to find the derivative in δ,

$$\partial_s S^\sigma \cdot [\partial_s S^{-\sigma}]_*^\perp = \partial_s S \cdot \partial_s S_*^\perp + \sigma\delta(\partial_s S \cdot T_* + T \cdot \partial_s S_*) + O(\delta^2).$$

Hence, its δ-derivative is some bounded function, say $E(s,s_*)$. When ∂_δ hits the kernel $|S^\sigma - [S^{-\sigma}]_*|^{-1-\alpha}$, we have

$$\begin{aligned}
&\partial_\delta |S^\sigma - [S^{-\sigma}]_*|^{-1-\alpha} \\
&= (-1-\alpha)|S^\sigma - [S^{-\sigma}]_*|^{-3-\alpha}(S^\sigma - [S^{-\sigma}]_*) \cdot \partial_\delta (S^\sigma - [S^{-\sigma}]_*) \\
&= (-1-\alpha)|S^\sigma - [S^{-\sigma}]_*|^{-3-\alpha}(S^\sigma - [S^{-\sigma}]_*) \cdot \sigma(N + N_*) \\
&= (-1-\alpha)|S^\sigma - [S^{-\sigma}]_*|^{-3-\alpha}[(S - S_*) \cdot \sigma(N + N_*) + O(\delta)].
\end{aligned}$$

With the cancellation from the symmetrisation in Γ, we see that we have

$$\begin{aligned}
-8\mathcal{B}'(\delta) = &\sum_{\sigma=\pm 1} \sigma \int_{\mathbb{T}}\int_{\mathbb{T}} |S^\sigma - [S^{-\sigma}]_*|^{-1-\alpha}(\Gamma(S) - \Gamma(S_*)) \\
&\times \left(E(s,s_*) + (-1-\alpha)\partial_s S \cdot [\partial_s S]_*^\perp \frac{S^\sigma - [S^{-\sigma}]_*}{|S^\sigma - [S^{-\sigma}]_*|^2} \right) \mathrm{d}s\, \mathrm{d}s_* + O(\delta).
\end{aligned}$$

The factor of σ means that we can use the Mean Value Theorem in the form $f(x+\delta) - f(x-\delta) = O(\delta)$ to obtain that actually $\mathcal{B}'(\delta) = O(\delta)$, which finally implies the result. $\square$

Chapter 7

Conclusions and Open Problems

7.1 Summary of results

In this thesis, we have studied sharp fronts and almost-sharp fronts of the singular generalisation of the SQG equation (1.1) which we now recall,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} |\nabla|^{-1+\alpha} \theta, \end{cases}$$

motivated by extrapolating the well-known PDEs at $\alpha = -1, 0$ to the more singular range $\alpha \in (0, 1)$. We have extended a number of results of Fefferman and Rodrigo on SQG sharp fronts and almost-sharp fronts with one periodic space variable to our singular SQG equation.

More specifically, we derived the sharp front equation,

$$z_t(s,t) \cdot N(s,t) = -\int_{\mathbb{T}} \frac{z_s(s_*,t) - z(s,t)}{|z(s_*,t) - z(s,t)|^{1+\alpha}} \, \mathrm{d}s_* \cdot N(s,t),$$

rigourously from the definition of a weak solution to SQG. The freedom in the choice of parameterisation was used to derive the equivalent formulation,

$$z_t(s,t) = -\int_{\mathbb{T}} \frac{z_s(s_*,t) - z(s,t)}{|z(s_*,t) - z(s,t)|^{1+\alpha}} \, \mathrm{d}s_* + \lambda(s,t) z_s(s,t).$$

We showed the local existence of solutions to this modified equation in the analytic setting by using the abstract Cauchy–Kowalevskaya theorem, which was possible despite the presence of an operator of order higher than one.

Then, we defined an almost-sharp front and its compatible curves, and derived an asymptotic equation that the almost-sharp front family must solve. Then we proved that the evolution of the compatible curve differs from that of a sharp front by an error of size $O(\delta^{1-\alpha})$.

Finally, we showed that the measure-theoretic spine construction of Fefferman, Luli and Rodrigo generalises to our setting, and allows us to select a special curve supported in the transition region of an almost-sharp front whose evolution more closely approximates the sharp front equation by a whole power of δ in the error.

7.2 Further research directions

In this section, we discuss some ideas related to this thesis that could form the basis of future work.

Existence of δ-almost-sharp fronts for times independent of δ

In [30], the authors prove the local existence and uniqueness of the δ-almost-sharp front family of solutions for SQG, in the class of analytic functions with a time of existence that does not depend on δ. This result was proven by finding a suitable limit equation, and studying a naturally defined object (the h function obtained by integrating across the transition region, which we also found in Section 5.3). Results of this nature can motivate the definition of a 'sharp front' even in situations where 'sharp fronts' are not natural or easy to study, like a vortex filament.

Actually, a number of the results in this thesis were proven in part to prepare for proving the analogous result for our singular SQG equation, and it would be very interesting if this plan could be followed through to completion. The main stumbling block seems to be from the fact that the approximate equation for singular SQG has a bad term with power-law dependence on δ. The logarithm present in the SQG almost-sharp front serves to separate (using the property $\log(ab) = \log a + \log b$) two bad effects which can be dealt with separately. To complete this line of proof, it seems that we would need to discover a natural coordinate system that can remove both the bad effects at once, or it could be that there is further structure in the approximate equation than just the h function that can be used to regularise the equation.

The almost-sharp fronts of SQG also seem to share some formal properties with the (2D) Prandtl equation, since roughly speaking, the asymptotic analysis of the almost-sharp front detected an imbalance in the number or derivatives in the limit as $\delta \to 0$. This is a similar situation to the Prandtl equation where there is a 'smoothing operator' ∂_y^{-1} in one direction but not the other. It may be possible to use similar techniques to the classic work of Oleinik [59], [58] to prove local existence of almost-sharp fronts in a class of monotone solutions. Furthermore, it is known [34] that the Prandtl equation is linearly ill-posed when linearised around a shear flow. It would be interesting if an analogue can be proven for our equation: it is currently not clear what would be the correct setting for stability analysis, or even numerical exploration.

Sharp fronts for logarithmically hypersingular kernels

From (1.1), sending $\alpha \to 1$ formally would seem to lead us to a degenerate equation,

$$0 = \theta_t + \nabla^{\perp}\theta \cdot \nabla\theta \iff \theta_t = 0.$$

However, from the work of Ohkitani [57], we see that we can introduce $\nabla^{\perp}\theta$ to reveal a finite difference in α (i.e. something of the form $f(a+\alpha) - f(a)$)

$$0 = \theta_t + (|\nabla|^{-1+\alpha}\nabla^{\perp}\theta - \nabla^{\perp}\theta) \cdot \nabla\theta.$$

Choosing a rescaling in time that depends on the parameter α,

$$t \mapsto (-1+\alpha)t,$$

we find a difference quotient with step size α, and now sending $\alpha \to 0$ formally gives the velocity

$$u = \log|\nabla|\nabla^{\perp}\theta.$$

The kernel of this operator is more singular than the one in (1.1), and it would be very interesting to understand in what sense the above can be made rigourous, and the solution theory of this equation needs to be developed. In the papers of Chae et al. [14],[13] inspired by Ohkitani's work, there are some results about very similar equations, but it is not clear if those equations are more natural than the one formally derived here, and they do not discuss the relationships between the models or any limit as $\alpha \to 0$. This related model is the active scalar transport equation with velocity

$$u = \log\big(1 + |\nabla|^2\big)\nabla^{\perp}\theta.$$

There, they proved existence of weak solutions in Hilbert based Sobolev spaces. Here, the convolution kernel for the operator $\log\big(1+|\nabla|^2\big)$ has the Fourier transform

$$\hat{K} = \frac{1}{|x|^2}\hat{f},$$

where $f(x) = \frac{c}{(1+|x|^2)^2}$ is a solution to a Liouville equation (namely, $\Delta \log f = cf$). The connection between these two equations may lead to a nice solution theory for this (and related) models of SQG that are more singular than those considered in this thesis.

Notably, they did not consider sharp fronts for this equation in [13], despite proving local existence for the sharp fronts for (1.1). It will be interesting to see if a theory of sharp front solutions can be built for this equation. It is a priori not obvious what the result should be, since the kernel is so singular. In particular, new techniques may be required to understand sharp fronts for this equation.

Survival of the spine curve beyond $\alpha = 1$

As noted, the evolution equation of the spine curve S matches the evolution equation of a sharp front, up to an error $O(\delta^{2-\alpha})$, which is not to be expected for a generic compatible curve. This error term is so small that it formally allows a velocity that is more singular than even the logarithmically hypersingular velocities discussed above. If for example, we can at least prove that smooth solutions exist for the equation (where $\beta > 0$)

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ \quad u = \nabla^\perp|\nabla|^\beta\theta, \end{cases}$$

then the behaviour of a spine curve for this equation with $\delta \ll 1$ would describe the only possible evolution of a sharp front, even if the equation is too badly behaved to derive a sharp front equation from the definition of a weak solution.

Fast dynamics: time-rescaling the almost-sharp front equation

The almost-sharp front equation of thickness δ for the model (1.1) after the following rescaling in time,

$$t_0 = t/\delta^\alpha, \quad \partial_{t_0} = \delta^\alpha\partial_t,$$

gives rise to a simple equation in the formal limit $\delta \to 0$,

$$\Omega_t + \mathcal{J}(\nabla^\perp\Omega)\cdot\nabla\Omega = 0.$$

Here, $\mathcal{J}$ is some smoothing operator that only acts in the vertical direction, similar to the appearance of ∂_y^{-1} in Prandtl without any smoothing in x. The time rescaling is reminiscent of the time-scaling needed to formally derive the local induction eqaution for vortex filaments. Given the simple structure of this equation, it should not be too hard to develop some basic theory for it. Similar types of solutions to the almost-sharp front should be obtainable, and it would be interesting to see if this equation exhibits linear instability in Sobolev spaces like Prandtl does, as shown in [34]. A complete theory for this equation may lead to a more complete understanding of the model (1.1) and related equations.

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